1

VON NEUMANN ALGEBRAS

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1. Review of functional analysis

In this section we state the results that we will need from functional analysis. All of these are stated and proved in [Fo99, Chapters 4-7].

Convention. All vector spaces considered below are over $\mathbb{C}$.

1.1. Normed vector spaces.

Definition 1.1. A **normed vector space** is a vector space X over $\mathbb{C}$ together with a map $\|\cdot\| : X \to [0, \infty)$ which is a **norm**, i.e., it satisfies that

- $\|x + y\| \leq \|x\| + \|y\|$, for al $x, y \in X$,
- $\|\alpha x\| = |\alpha| \, \|x\|$, for all $x \in X$ and $\alpha \in \mathbb{C}$, and
- $\|x\| = 0 \Leftrightarrow x = 0$, for all $x \in X$.

Definition 1.2. Let X be a normed vector space.

(1) A map $\varphi : X \to \mathbb{C}$ is called a **linear functional** if it satisfies $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$, for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$. A linear functional $\varphi : X \to \mathbb{C}$ is called **bounded** if $\|\varphi\| := \sup_{\|x\|\leq 1} |\varphi(x)| < \infty$. The **dual of** X, denoted X^*, is the normed vector space of all bounded linear functionals $\varphi : X \to \mathbb{C}$.

(2) A map $T : X \to X$ is called **linear** if it satisfies $T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)$, for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$. A linear map $T : X \to X$ is called **bounded** if $\|T\| := \sup_{\|x\|\leq 1} \|T(x)\| < \infty$. A linear bounded map T is usually called a linear bounded operator, or simply a bounded operator. We denote by $\mathbb{B}(X)$ the normed vector space of all bounded operators $T : X \to X$.

Definition 1.3. A **Banach space** is a normed vector space X which is **complete** in the norm metric: any Cauchy sequence $\{x_n\}$ (i.e., such that $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$) is convergent (i.e., there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$).

Examples 1.4. (of Banach spaces):

- $\mathbb{C}^n$, for $n \geq 1$, with the Euclidean norm $\|(x_1, ..., x_n)\| = \sqrt{x_1^2 + ... + x_n^2}$,
- $L^p(Y)$, for any measure space (Y, μ) and $1 \leq p \leq \infty$, with the L^p-norm: $$\|f\|_p = \begin{cases} (\int |f|^p)^{1/p}, \text{ if } p \text{ is finite,} \\ \inf\{\alpha > 0 \mid |f(y)| \leq \alpha, \text{ for } \mu\text{-almost every } y \in Y\}, \text{ if } p = \infty. \end{cases}$$
- $\ell^p(I)$, for any set I and $1 \leq p \leq \infty$, with the ℓ^p-norm: $\|f\|_p = \begin{cases} (\sum_{i\in I} |f(i)|^p)^{1/p}, \text{ if } p \text{ is finite,} \\ \sup_{i\in I} |f(i)|, \text{ if } p = \infty. \end{cases}$
- $C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous}\}$, for any compact Hausdorff topological space X, with the supremum norm $\|f\| = \sup_{x\in X} |f(x)|$.

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- $B(X) = \{f : X \to \mathbb{C} \mid f \text{ bounded Borel}\}$, for any compact Hausdorff topological space X, with the supremum norm $\|f\| = \sup_{x\in X} |f(x)|$.
- The dual X^* of any normed vector space X.
(Recall that $L^p(Y)^* = L^q(Y)$ and $\ell^p(I)^* = \ell^q(I)$, if $1 \leq p < \infty$ and $1/p + 1/q = 1$.)
- $\mathbb{B}(X)$, for any Banach space X.

Definition 1.5. A **Hilbert space** H is a Banach space whose norm comes from a scalar product, i.e., there exists a scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that $\|x\| = \sqrt{\langle x, x\rangle}$, for all $x \in H$.

Examples 1.6. (of Hilbert spaces):

- $\mathbb{C}^n$, for $n \geq 1$, with the Euclidean scalar product $\langle (x_1, ..., x_n), (y_1, ..., y_n)\rangle = x_1\bar{y}_1 + ... + x_n\bar{y}_n$.
- $L^2(Y)$, for any measure space (Y, μ), with the scalar product $\langle f, g\rangle = \int f\bar{g}$.
- $\ell^2(I)$, for any set I, with the scalar product $\langle f, g\rangle = \sum_{i\in I} f(i)\overline{g(i)}$.
- Recall that every Hilbert space H has an **orthonormal basis**, i.e., a set $\{\xi_i\}_{i\in I}$ such that $\langle \xi_i, \xi_j\rangle = \delta_{i,j}$, for all $i, j \in I$, and $\xi = \sum_{i\in I}\langle \xi, \xi_i\rangle\xi_i$, for all $\xi \in H$. This fact implies that every Hilbert space H is isomorphic to $\ell^2(I)$, for some set I.

1.2. **Linear functionals.**

Theorem 1.7 (Hahn-Banach)**.** *Let X be a normed vector space, $Y \subset X$ a subspace, and $\varphi : Y \to \mathbb{C}$ a linear functional such that $|\varphi(x)| \leq \|x\|$, for all $x \in Y$. Then there exists $\tilde{\varphi} \in X^*$ such that $|\tilde{\varphi}(x)| \leq \|x\|$, for all $x \in X$, and $\tilde{\varphi}_{|Y} = \varphi$.*

Definition 1.8. Let X be a normed vector space. The **weak* topology** on X^* is the topology of pointwise convergence: $\varphi_i \to \varphi$ iff $\varphi_i(x) \to \varphi(x)$, for all $x \in X$. The **weak topology** on X is given by $x_i \to x$ iff $\varphi(x_i) \to \varphi(x)$, for all $\varphi \in X^*$.

Theorem 1.9 (Alaoglu)**.** *The closed unit ball of X^*, $B := \{\varphi \in X^* \mid \|\varphi\| \leq 1\}$, is compact in the weak* topology.*

Proof. Put $D_x = \{z \in \mathbb{C} \mid |z| \leq \|x\|\}$ for $x \in X$. Then $P := \prod_{x\in X} D_x$ is compact by Tychonoff's theorem. Define $\theta : B \to P$ by letting $\theta(\varphi) = (\varphi(x))_{x\in X}$. Then $\varphi_i \to \varphi$ is the weak* topology iff $\theta(\varphi_i) \to \theta(\varphi)$ in the product topology. Thus, in order to show that B is compact in the weak* topology (equivalently, that every net $(\varphi_i) \in B$ has a weak* convergent subnet) it suffices to prove that $\theta(B)$ is compact in the product topology. As it can be easily seen that $\theta(B)$ is closed, and thus compact, in the product topology, the conclusion follows. ∎

Theorem 1.10. *Let H be a Hilbert space. If $\varphi \in H^*$, then there exists $y \in H$ such that*

$$\varphi(x) = \langle x, y\rangle, \text{ for all } x \in H.$$

Proof. Let $K = \{x \in H \mid \varphi(x) = 0\}$. Since φ is bounded, K is a closed subspace of H. If $K = H$, then $\varphi \equiv 0$ and $y = 0$ works. If K is a proper subspace of H, we can find $z \in H$ such that $z \perp K$ and $\|z\| = 1$ (see [Fo99, Theorem 5.24]). Since $\varphi(\varphi(z) \cdot x - \varphi(x) \cdot z) = 0$, we have $\varphi(z) \cdot x - \varphi(x) \cdot z \in K$, hence $\langle \varphi(z) \cdot x, z\rangle = \langle \varphi(x) \cdot z, z\rangle$, thus $\varphi(x) = \langle x, \overline{\varphi(z)}z\rangle$, for all $x \in H$. ∎

Definition 1.11. Let X be a compact Hausdorff topological space. A **Borel measure** on X is a measure defined on the σ-algebra $\mathcal{B}_X$ of all Borel subsets of X. A finite Borel measure μ on X is called **regular** if $\mu(A) = \inf\{\mu(U) \mid U \supset A \text{ open}\} = \sup\{\mu(K) \mid K \subset A \text{ compact}\}$, for every Borel set $A \subset X$. A **complex Borel regular measure** on X is a map $\mu : \mathcal{B}_X \to \mathbb{C}$ of the form $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$, where $\mu_1, ..., \mu_4$ are finite Borel regular measures on X.

Theorem 1.12 (Riesz's Representation Theorem)**.** *Let X be a compact Hausdorff topological space. If $\varphi \in C(X)^*$, then there exists a complex Borel regular measure μ on X such that $\varphi(f) = \int f \, d\mu$. Moreover, we have that $\|\varphi\| = \|\mu\| := \sup\{\sum_{i=1}^n |\mu(A_i)| \mid \{A_i\}_{i=1}^n \text{ Borel partition of } X\}$.*

Theorem 1.13 (Stone-Weierstrass)**.** *Let X be a compact Hausdorff topological space. Let A be a closed subalgebra of $C(X)$ such that*

- $1_X \in A$,
- *if $f \in A$ then $\bar{f} \in A$, and*
- *A separates points: for all $x \neq y \in X$, there is $f \in A$ such that $f(x) \neq f(y)$.*

Then $A = C(X)$.

1.3. **The adjoint operation and topologies on $\mathbb{B}(H)$.** Let H be a Hilbert space.

Exercise 1.14. Let $T \in \mathbb{B}(H)$. Prove that there exists a unique $T^* \in \mathbb{B}(H)$, called the **adjoint** of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x, y \in H$. Prove that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$.

Definition 1.15. An operator $T \in \mathbb{B}(H)$ is called:

- **self-adjoint** (or, **hermitian**) if $T^* = T$,
- a **projection** if $T = T^* = T^2$,
- a **unitary** if $T^*T = TT^* = \mathrm{Id}_H$,
- an **isometry** if $\|Tx\| = \|x\|$, for all $x \in H$, or equivalently $T^*T = \mathrm{Id}_H$,
- **normal** if $T^*T = TT^*$,
- **compact** if the closure of $\{T(x) \mid \|x\| \leq 1\}$ in H is compact.

Remark 1.16. The set of unitary operators $T \in \mathbb{B}(H)$ is group which is denoted by $\mathcal{U}(H)$.

Exercise 1.17. Let $T \in \mathbb{B}(H)$ be a projection. Prove that there exists a closed subspace $K \subset H$ such that T is the **orthogonal projection** onto K (see [Fo99, page 177, Exercise 58]).

Definition 1.18. There are three topologies on $\mathbb{B}(H)$ that we will consider:

- the **norm topology**: $T_i \to T$ iff $\|T_i - T\| \to 0$.
- the **strong operator topology (SOT)**: $T_i \to T$ iff $\|T_i(\xi) - T(\xi)\| \to 0$, for all $\xi \in H$.
- the **weak operator topology (WOT)**: $T_i \to T$ iff $\langle T_i(\xi), \eta\rangle \to \langle T(\xi), \eta\rangle$, for all $\xi, \eta \in H$.

Exercise 1.19. Let $(T_i)_{i\in I} \subset \mathbb{B}(H)$ be a net such that $T_i \to T$ (WOT), for some $T \in \mathbb{B}(H)$.

(1) Assume that $(T_i)_{i\in I}$ and T are projections. Prove that $T_i \to T$ (SOT).
(2) Assume that $(T_i)_{i\in I}$ and T are unitaries. Prove that $T_i \to T$ (SOT).

Exercise 1.20. Assume that H is infinite dimensional ($\Longleftrightarrow$ any orthonormal basis of H is infinite).

(1) Given an example of a net of projections $(T_i)_{i\in I}$ converging in the WOT but not the SOT.
(2) Given an example of a net of unitaries $(T_i)_{i\in I}$ converging in the WOT but not the SOT.

Exercise 1.21. Prove that the closed unit ball of $\mathbb{B}(H)$, $B := \{T \in \mathbb{B}(H) \mid \|T\| \leq 1\}$, is compact in the WOT.

Proposition 1.22. *If $C \subset \mathbb{B}(H)$ is a convex set, then $\overline{C}^{SOT} = \overline{C}^{WOT}$.*

The proof of Proposition 1.22 relies on the following lemma:

Lemma 1.23. *If $C \subset H$ is a convex set, then the weak and norm closures of C are equal.*

Proof. By Theorem 1.10 the weak topology on H is given by: $\xi_i \to \xi$ **weakly** iff $\langle \xi_i, \eta\rangle \to \langle \xi, \eta\rangle$, for all $\eta \in H$. Denote $D = \overline{C}^{\|\cdot\|}$. It is clear that $D \subset \overline{C}^{weak}$. To prove the reverse inclusion, let $\xi \in \overline{C}^{weak}$. Since D is a norm closed and convex subset of the Hilbert space H, we can find $\eta_0 \in D$ such that $\|\xi - \eta_0\| = \inf_{\eta\in D} \|\xi - \eta\|$ (see [Fo99, page 177, Exercise 58]). Let $\eta \in C$. Then the function $[0,1] \ni t \to \|\xi-(1-t)\eta_0-t\eta\|^2 = \|(\xi-\eta_0)-t(\eta-\eta_0)\|^2$ has a minimum at $t=0$, hence its derivative at $t=0$ is positive. It follows that $\Re\langle \xi-\eta_0, \eta-\eta_0\rangle \leqslant 0$, for all $\eta \in C$. Finally, let $\eta_i \in C$ be a net such that $\eta_i \to \xi$ weakly. We get that $\Re\langle \xi-\eta_0, \xi-\eta_0\rangle \leqslant 0$ and therefore $\xi = \eta_0 \in D$. ■

Proof of Proposition 1.22. Let $y \in \overline{C}^{WOT}$, $\xi_1, ..., \xi_n \in H$ and $\varepsilon > 0$. Then $D = \{(x\xi_1, ..., x\xi_n) | x \in C\}$ is a convex subset of $H^n = \oplus_{i=1}^n H$. Since $(y\xi_1, ..., y\xi_n)$ is in the weak closure of D, by Lemma 1.23 it is also in the norm closure of D. Therefore, we can find $x \in C$ such that $(\sum_{i=1}^n \|x\xi_i - y\xi_i\|^2)^{1/2} < \varepsilon$. This implies that $y \in \overline{C}^{SOT}$. Since the inclusion $\overline{C}^{SOT} \subset \overline{C}^{WOT}$ also holds, we are done. ■

2. von Neumann algebras basics

Definition 2.1. Let H be a Hilbert space.

- A subalgebra $A \subset \mathbb{B}(H)$ is called a **$*$-algebra** if $T^* \in A$, for every $T \in A$.
- A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a [1] **C*-algebra** if it closed in the norm topology.
- A $*$-subalgebra $A \subset \mathbb{B}(H)$ is called a **von Neumann algebra** if it is WOT-closed.

Definition 2.2. A map $\pi : A \to B$ between two C*-algebras is a **$*$-homomorphism** if it is a homomorphism ($\pi(a+b) = \pi(a) + \pi(b)$, $\pi(ab) = \pi(a)\pi(b)$, $\pi(\lambda a) = \lambda\pi(a)$, for all $a, b \in A$, $\lambda \in \mathbb{C}$) and satisfies $\pi(a^*) = \pi(a)^*$ for all $a \in A$. A bijective $*$-homomorphism is called a **$*$-isomorphism**.

Examples 2.3. (of C*-algebras and von Neumann algebras):

(1) Any von Neumann algebra is a C*-algebra.
(2) $\mathbb{B}(H)$ is a von Neumann algebra.
(3) $\mathbb{K}(H)$, the algebra of compact operators on H, is a C*-algebra.
(4) Let $B \subset \mathbb{B}(H)$ be a set such that $T^* \in B$, for every $T \in B$. Then the **commutant** of B, defined as $B' = \{T \in \mathbb{B}(H) \mid TS = ST, \text{ for every } S \in B\}$ is a von Neumann algebra.

Conversely, the next theorem shows that every von Neumann algebra arises this way.

Theorem 2.4 (von Neumann's bicommutant theorem)**.** *If $M \subset \mathbb{B}(H)$ is a unital $*$-subalgebra, then the following three conditions are equivalent:*

(1) M is WOT-closed.
(2) M is SOT-closed.
(3) $M = M'' := (M')'$.

This is a beautiful result which asserts that for $*$-algebras, the analytic condition of being closed in the WOT is equivalent to the algebraic condition of being equal to their double commutant.

Remark 2.5. Let $S \subset \mathbb{B}(H)$ be a $*$-set which contains the identity. By Theorem 2.4, S'' is equal to the von Neumann algebra generated by S, i.e., the smallest von Neumann algebra containing S.

Proof. It is clear that (3) $\Rightarrow$ (1) $\Rightarrow$ (2). To prove that (2) $\Rightarrow$ (3), it suffices to show that if $x \in M''$, $\varepsilon > 0$, and $\xi_1, ..., \xi_n \in H$, then there exists $y \in M$ such that $\|x\xi_i - y\xi_i\| < \varepsilon$, for all $i = 1, ..., n$.

We start with the following claim. Let p be the orthogonal projection from H onto an M-invariant closed subspace K. Then $p \in M'$. To see this, let $x \in M$. Then $(1-p)xp\xi \in (1-p)(K) = \{0\}$, for all $\xi \in H$. Hence $(1-p)xp = 0$ and so $xp = pxp$. By taking adjoints, we get that $px^* = px^*p$ and hence $px = pxp$, for all $x \in M$. This shows that p commutes with x, as claimed.

[1] Strictly speaking, this defines a C*algebra that is *concrete*, i.e., one which is represented concretely on a Hilbert space. We will later define what an *abstract* C*-algebra means and see that the two notions are equivalent.

Assume first that $n = 1$ and let p be the orthogonal projection onto $\overline{M\xi_1} = \overline{\{x\xi_1 \mid x \in M\}}$. Since $\overline{M\xi_1}$ is M-invariant, Claim 1 gives that $p \in M'$. Thus $xp = px$ and so $x\xi_1 = xp\xi_1 = px\xi_1 \in \overline{M\xi_1}$. Therefore, there is $y \in M$ such that $\|x\xi_1 - y\xi_1\| < \varepsilon$.

Now, for arbitrary $n \geq 2$, we use a "matrix trick". Let $H^n = \oplus_{i=1}^n H$ be the direct sum of n copies of H and identify $\mathbb{B}(H^n) = \mathbb{M}_n(\mathbb{B}(H))$. Let $\pi : M \to \mathbb{B}(H^n)$ be the "diagonal" $*$-representation

$$\pi(a) = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a \end{pmatrix}$$

Equivalently, $\pi(a)(\xi_1 \oplus \dots \oplus \xi_n) = a\xi_1 \oplus \dots \oplus a\xi_n$.

Exercise 2.6. Prove that the following holds: $\pi(M'') \subset \mathbb{M}_n(M')'$ and $\pi(M)' \subset \mathbb{M}_n(M')$.

Finally, if $x \in M''$, then Exercise 2.6 gives that $\pi(x) \in \pi(M)''$. Let $\xi = (\xi_1, \dots, \xi_n) \in H^n$. By applying the case $n = 1$ we conclude that there is $y \in M$ such that $\|\pi(x)\xi - \pi(y)\xi\| < \varepsilon$. Since $\|\pi(x)\xi - \pi(y)\xi\|^2 = \sum_{i=1}^n \|x\xi_i - y\xi_i\|^2$, we are done. ■

Definition 2.7. A **Polish space** is a topological space X which is metrizable, complete and separable. A measure space (X, μ) is called a **probability space** if $\mu(X) = 1$. A probability space (X, μ) is called **standard** if X is a Polish space and μ is a Borel probability measure on X.

Proposition 2.8. *Let (X, μ) be a standard probability space. Define $\pi : L^\infty(X, \mu) \to \mathbb{B}(L^2(X, \mu))$ by letting $\pi_f(\xi) = f\xi$, for all $f \in L^\infty(X)$ and $\xi \in L^2(X)$. Then $\pi(L^\infty(X))' = \pi(L^\infty(X))$. Therefore, $\pi(L^\infty(X)) \subset \mathbb{B}(L^2(X))$ is a maximal abelian von Neumann subalgebra.*

Proof. Let $T \in \pi(L^\infty(X))'$ and put $g = T(1)$. Then $fg = \pi_f T(1) = T\pi_f(1) = T(f)$ and hence

$$\|fg\|_2 = \|T(f)\|_2 \leqslant \|T\| \, \|f\|_2, \quad \text{for every} \quad f \in L^\infty(X).$$

Let $\varepsilon > 0$ and $f = 1_{\{x \in X \mid |g(x)| \geqslant \|T\| + \varepsilon\}}$. Then it is clear that $\|fg\|_2 \geqslant (\|T\| + \varepsilon)\|f\|_2$. In combination with the last inequality, we get that $(\|T\| + \varepsilon)\|f\|_2 \leqslant \|T\|\|f\|_2$, and so $f = 0$, almost everywhere. Thus, we conclude that $g \in L^\infty(X)$. Since $T(f) = fg = \pi_g(f)$, for all $f \in L^\infty(X)$, and $L^\infty(X)$ is $\|.\|_2$-dense in $L^2(X)$, it follows that $T = \pi_g \in L^\infty(X)$. ■

Exercise 2.9. Let I be a set. Define $\pi : \ell^\infty(I) \to \mathbb{B}(\ell^2(I))$ by $\pi_f(g)(i) = f(i)g(i)$, for all $i \in I$, $f \in \ell^\infty(I)$ and $g \in \ell^2(I)$. Prove that $\pi(\ell^\infty(I))' = \pi(\ell^\infty(I))$. (Therefore, $\pi(\ell^\infty(I)) \subset \mathbb{B}(\ell^2(I))$ is a maximal abelian von Neumann subalgebra.)

3. Abelian group algebras and standard probability spaces

3.1. **Group von Neumann algebras.** Let Γ be a countable group. A **unitary representation** of Γ on a Hilbert space H is a group homomorphism $\pi : \Gamma \to \mathcal{U}(H)$. Every countable group Γ has a canonical unitary representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ called the **left regular representation** and defined by $\lambda(g)f(h) = f(g^{-1}h)$, for all $g, h \in \Gamma$ and $f \in \ell^2(\Gamma)$.

Remark 3.1. Let $\delta_g \in \ell^2(\Gamma)$ denote the Dirac mass at $g \in \Gamma$ given by $\delta_g(h) = \delta_{g,h}$. Then $\{\delta_g\}_{g \in \Gamma}$ is an orthonormal basis of $\ell^2(\Gamma)$, and we have $\lambda(g)(\delta_h) = \delta_{gh}$, for all $g, h \in \Gamma$.

Let $\mathcal{A} = \{\sum_{g \in F} a_g \lambda(g) \mid F \subset \Gamma \text{ finite}, \ a_g \in \mathbb{C}, \text{ for all } g \in F\}$. Then $\mathcal{A} \subset \mathbb{B}(\ell^2(\Gamma))$ is a $*$-subalgebra which is isomorphic to the complex group algebra $\mathbb{C}[\Gamma]$.

Definition 3.2. The **reduced C*-algebra** of Γ is defined as $C_r^*(\Gamma) := \overline{\mathcal{A}}^{\|\cdot\|}$.
The **group von Neumann algebra** of Γ is defined as $L(\Gamma) := \overline{\mathcal{A}}^{WOT}$.

The structure of $L(\Gamma)$ can be understood is a simpler way for abelian groups. Recall that the **Pontryagin dual** of a countable abelian group Γ, denoted by $\hat{\Gamma}$, consists of all **characters** of Γ, i.e., homomorphisms $h : \Gamma \to \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $\hat{\Gamma}$ is a compact metrizable group, when endowed with the topology of pointwise convergence: $h_i \to h$ iff $h_i(g) \to h(g)$, for all $g \in \Gamma$. Indeed, if we enumerate $\Gamma = \{g_n\}_{n\geqslant 1}$, then $d(h, h') = \sum_{n=1}^{\infty} 2^{-n}|h(g_n) - h'(g_n)|$ is a compatible metric.

Proposition 3.3. *Let Γ be a countable abelian group. Denote by μ the Haar measure of $\hat{\Gamma}$. Then $L(\Gamma)$ is $*$-isomorphic to $L^\infty(\hat{\Gamma}, \mu)$, and $C_r^*(\Gamma)$ is $*$-isomorphic to $C(\hat{\Gamma})$.*

Before proving Proposition 3.3, we establish the following useful fact (see also [Co99, Lemma 9.7]):

Lemma 3.4. *Let X be a topological space which is* **normal** *in the sense that any two disjoint closed sets have disjoint open neighborhoods. If $f \in B(X)$, then there exists a net $f_i \in C(X)$ such that $\sup_i \|f_i\|_\infty \leqslant \|f\|_\infty$ and $\int_X f_i \, d\mu \to \int_X f \, d\mu$, for every Borel regular probability measure μ on X.*

Remark 3.5. Every compact Hausdorff space X is normal.

Proof. Let $f \in B(X)$, $\mu_1, ..., \mu_n$ be regular Borel probability measures on X, and $\varepsilon > 0$. Let $\alpha_1, ..., \alpha_m \in \mathbb{C}$ and $\Delta_1, ..., \Delta_m$ be disjoint Borel subsets of X such that $\|f - \sum_{k=1}^m \alpha_k 1_{\Delta_k}\|_\infty < \varepsilon/2$ and $|\alpha_k| \leqslant \|f\|_\infty$, for all $k = 1, ..., m$. Since $\mu_1, ..., \mu_n$ are regular, we can find closed sets $F_k \subset \Delta_k$ and open sets $G_k \supset \Delta_k$ such that $\mu_i(G_k \setminus F_k) < \varepsilon/(4m\|f\|_\infty)$, for all $k = 1, ..., m$ and $i = 1, ..., n$. Thus, if $F = \cup_{k=1}^m F_k$ and $G = \cup_{k=1}^m G_k$, then $\mu_i(G \setminus F) < \varepsilon/(4\|f\|_\infty)$, for all $i = 1, ..., n$.

Let $Y := F \cup (X \setminus G)$ and define $h : Y \to \mathbb{C}$ by letting $h(x) = \alpha_k$ if $x \in F_k$ and $h(x) = 0$ if $x \notin G$. Then $Y \subset X$ is closed, $h \in C(Y)$, and $\|h\|_\infty \leq \|f\|_\infty$. Since X is normal, the Tietze Extension Theorem (see [Fo99, Theorem 4.16 and Corollary 4.17]) provides $\tilde{f} \in C(Y)$ such that $\|\tilde{f}\|_\infty \leq \|f\|_\infty$ and $\tilde{f}_{|Y} = h$. Then it is clear that $|\tilde{f} - \sum_{k=1}^m \alpha_k 1_{\Delta_k}| \leq 2\|f\|_\infty 1_{G\setminus F}$, and thus we have that

$$|\int_X \tilde{f} \, \mathrm{d}\mu_i - \int_X f \, \mathrm{d}\mu_i| \leqslant 2\|f\|_\infty \mu_i(G \setminus F) + \|f - \sum_{k=1}^m \alpha_k 1_{\Delta_k}\|_\infty \leqslant \varepsilon.$$

Since $\varepsilon > 0$ and the probability measures $\mu_1, ..., \mu_n$ are arbitrary, it is easy to finish the proof. ■

Lemma 3.6. *Let Γ be a countable abelian group. If $g \in \Gamma \setminus \{e\}$, then $h(g) \neq 1$ for some $h \in \hat{\Gamma}$.*

Proof. Enumerate $\Gamma = \{g_n\}_{n\geqslant 1}$ with $g_1 = g$. For any n, let $\Gamma_n < \Gamma$ be the subgroup generated by $\{g_1, ..., g_n\}$. Since $g \neq e$, there is a character $h_1 : \Gamma_1 \to \mathbb{T}$ such that $h_1(g) \neq 1$. We prove by induction that there is a character $h_n : \Gamma_n \to \mathbb{T}$ such that $h_{n+1|\Gamma_n} = h_n$, for all $n \geqslant 1$. Once this is done, it is clear how to define h. Thus, it suffices to show that a character h_n of Γ_n extends to a character of Γ_{n+1}. Indeed, let l be the smallest integer such that $g_{n+1}^l \in \Gamma_n$. Define $h_{n+1}(g_{n+1}) = z$, where $z \in \mathbb{T}$ is such that $z^l = h_n(g_{n+1}^l)$. Then h_{n+1} has the desired property. ■

Proof of Proposition 3.3. For $g \in \Gamma$, let $\hat{g} \in L^2(\hat{\Gamma})$ be given by $\hat{g}(h) = h(g)$. If $g \in \Gamma \setminus \{e\}$, Lemma 3.6 gives $h' \in \hat{\Gamma}$ such that $h'(g) \neq 1$. Then

$$\int_{\hat{\Gamma}} \hat{g}(h) \, \mathrm{d}\mu(h) = \int_{\hat{\Gamma}} \hat{g}(h'h) \, \mathrm{d}\mu(h) = h'(g) \int_{\hat{\Gamma}} \hat{g}(h) \, \mathrm{d}\mu(h),$$

hence $\int_{\hat{\Gamma}} \hat{g}(h) \, \mathrm{d}\mu(h) = 0$. Thus, the formula $U(\delta_g) = \hat{g}$ defines an isometry $U : \ell^2(\Gamma) \to L^2(\hat{\Gamma})$.

We claim that U is onto, i.e., U is a unitary. Denote by $\mathcal{B}$ the linear span of $\{\hat{g} \mid g \in \Gamma\}$. Since $\mathcal{B}$ separates points in $\hat{\Gamma}$, the Stone-Weierstrass theorem implies that $\mathcal{B}$ is $\|.\|_\infty$-dense in $C(\hat{\Gamma})$. Since $\hat{\Gamma}$ is a compact Hausdorff space, Lemma 3.4 immediately implies that $C(\hat{\Gamma})$ is $\|.\|_2$-dense in $L^2(\hat{\Gamma})$. By combining these two facts, we conclude that U is onto.

Let $\pi : L^\infty(\hat{\Gamma}) \to \mathbb{B}(L^2(\hat{\Gamma}))$ be given as in Proposition 2.8. Then it is clear that $U\lambda(g)U^* = \pi_{\hat{g}}$, for all $g \in \Gamma$. If $\mathcal{A}$ denotes the linear span of $\{\lambda(g) \mid g \in \Gamma\}$, then $U\mathcal{A}U^* = \pi(\mathcal{B})$. Since π is isometric and the inclusions $\mathcal{A} \subset C_r^*(\Gamma)$ and $\mathcal{B} \subset C(\hat{\Gamma})$ are norm dense, we get that $UC_r^*(\Gamma)U^* = \pi(C(\hat{\Gamma}))$.

We claim that $\pi(C(\hat{\Gamma}))$ is WOT-dense in $\pi(L^\infty(\hat{\Gamma}))$. This will imply that $UL(\Gamma)U^* = \pi(L^\infty(\hat{\Gamma}))$, and finish the proof. To prove the claim, let $f \in L^\infty(\hat{\Gamma})$. After changing f on a set of measure zero, we may assume that $f \in B(\hat{\Gamma})$. Since $\hat{\Gamma}$ is a compact Hausdorff space, by applying Lemma 3.4 we can find a net $f_i \in C(\hat{\Gamma})$ such that $\int_{\hat{\Gamma}} f_i \, \mathrm{d}\mu \to \int_{\hat{\Gamma}} f \, \mathrm{d}\mu$, for every regular Borel measure μ on $\hat{\Gamma}$. Then for every $\xi, \eta \in L^2(\hat{\Gamma})$, we have $\langle \pi_{f_i}(\xi), \eta\rangle = \int_{\hat{\Gamma}} f_i \, (\xi\bar{\eta}) \, \mathrm{d}\mu \to \int_{\hat{\Gamma}} f \, (\xi\bar{\eta}) \, \mathrm{d}\mu = \langle \pi_f(\xi), \eta\rangle$. This shows that $\pi_{f_i} \to \pi_f$ in the WOT and proves our claim. ■

3.2. **Standard probability spaces.** We next prove that any non-atomic standard probability space is isomorphic to $([0,1], \lambda)$. More precisely, we have:

Theorem 3.7. *Let (X, μ) be a standard probability space without atoms: $\mu(\{x\}) = 0$, for every $x \in X$. Denote by λ be Lebesgue measure on $[0,1]$. Then there exist Borel sets $X_0 \subset X$, $Y_0 \subset [0,1]$ such that $\mu(X \setminus X_0) = \lambda([0,1] \setminus Y_0) = 0$, and a bijection $\theta : X_0 \to Y_0$ such that θ, θ^{-1} are Borel maps and $\theta_*\mu = \lambda$ (i.e. $\mu(\theta^{-1}(A)) = \lambda(A)$, for every Borel set $A \subset Y_0$).*

As a corollary, $\pi : L^\infty([0,1], \lambda) \to L^\infty(X, \mu)$ given by $\pi(f) = f \circ \theta$ is a $*$-isomorphism.

Remark 3.8. Theorem 3.7 holds in fact for $X_0 = X$ and $Y_0 = [0,1]$ (see [Ke95, Theorem 17.41]).

Lemma 3.9. *Let X be a Polish space. Then X is homeomorphic to a G_δ subset of $[0,1]^{\mathbb{N}}$. Here, $[0,1]^{\mathbb{N}}$ is endowed with the complete separable metric $d'((y_n)_n, (z_n)_n) = \sum_{n=1}^\infty 2^{-n}|y_n - z_n|$.*

Proof. Let d be a complete separable metric on X such that $d \leqslant 1$. Let $\{x_n\}$ be a dense sequence in X in which every element is repeated infinitely many times. Define $f : X \to [0,1]^{\mathbb{N}}$ by letting $f(x) = (d(x, x_n))_n$. Then f is injective and continuous. Moreover, $f^{-1}_{|f(X)}$ is continuous.

Indeed, let $x^k \in X$ be a sequence such that $f(x^k) \to f(x)$, for some $x \in X$. Then $\lim_{k\to\infty} d(x^k, x_n) = d(x, x_n)$, for all n. Fix $\varepsilon > 0$ and let n such that $d(x, x_n) < \varepsilon/2$. Then we can find K such that $d(x^k, x_n) < \varepsilon/2$ and hence $d(x^k, x) < \varepsilon$, for all $k \geqslant K$. This implies that $\{x^k\}$ converges to x.

Altogether, we conclude that X is homeomorphic to $f(X)$. To show that $f(X)$ is a G_δ set, we denote $U_p := \{y \in [0,1]^{\mathbb{N}} | \exists x \in X \text{ such that } d'(y, f(x)) < \frac{1}{p}\}$. Then U_p is open, for all p. Let V be the set of $y = (y_n)_n \in [0,1]^{\mathbb{N}}$ such that $d(x_m, x_n) \leqslant y_m + y_n$, for all $m, n \geqslant 1$. Since we can write $V = \cap_{m,n,N\geqslant 1}\{y | \; y_m + y_n > d(x_m, x_n) - \frac{1}{N}\}$, we get that V is a G_δ subset of $[0,1]^{\mathbb{N}}$. Also, let W be the set of $y = (y_n)_n \in [0,1]^{\mathbb{N}}$ such that $\liminf_{n\to\infty} y_n = 0$. Since $W = \cap_{m,N\geqslant 1}(\cup_{n\geqslant N}\{y | \; y_n < \frac{1}{m}\})$, we get that W is a G_δ subset of $[0,1]^{\mathbb{N}}$.

We claim that $f(X) = (\cap_{p\geqslant 1}U_p) \cap V \cap W$. Since the inclusion " $\subset$ " is clear, we only need to prove the reverse inclusion. Let $y = (y_n)_n \in (\cap_{p\geqslant 1}U_p) \cap V \cap W$. Since $y \in \cap_{p\geqslant 1}U_p$, there is a sequence $x^k \in X$ such that $\lim_{k\to\infty} d'(f(x^k), y) = 0$. Thus, $\lim_{k\to\infty} d(x^k, x_n) = y_n$, for all n. Since $y \in W$, we can find a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} < 2^{-i}$, for all i. Thus, for all i, we can find $k_i \geqslant 1$ such that $d(x^{k_i}, x_{n_i}) < 2^{-i}$. Since $y \in V$, it follows that the sequence $\{x_{n_i}\}$ is Cauchy, hence convergent.

Let $x = \lim_{i\to\infty} x_{n_i}$. Then we clearly get that $\lim_{i\to\infty} x^{k_i} = x$ and further that $y = f(x) \in f(X)$. This shows that $f(X)$ is G_δ and finishes the proof. ■

Proof of Theorem 3.7. Two Borel spaces X, Y are **Borel isomorphic** if there exists a bijection $\psi : X \to Y$ such that ψ and ψ^{-1} are Borel maps. Note that the dyadic expansion yields that $[0,1]$ is Borel isomorphic to $\{0,1\}^{\mathbb{N}}$. This fact implies the following Borel space are Borel isomorphic: $[0,1]^{\mathbb{N}} \cong (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \equiv \{0,1\}^{\mathbb{N}\times\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \cong [0,1]$. By Lemma 3.9, X is Borel isomorphic to a Borel subset of $[0,1]^{\mathbb{N}}$. By combining the last two facts we get that X is Borel isomorphic to a Borel subset of $[0,1]$. We may therefore assume that X is a Borel subset of $[0,1]$.

Next, we extend μ to a probability measure $\tilde{\mu}$ on $[0,1]$ by letting $\tilde{\mu}(A) = \mu(A \cap X)$, for every Borel set $A \subset X$. Define $\phi : [0,1] \to [0,1]$ by letting $\phi(x) = \tilde{\mu}([0,x])$. Then ϕ is increasing, continuous (since μ has not atoms), $\phi(0) = 0$ and $\phi(1) = 1$. Let $y \in [0,1]$ and let $x \in [0,1]$ be the largest number such that $\phi(x) = y$. Then we have that

$$\phi_*\tilde{\mu}([0,y]) = \tilde{\mu}(\{z \in [0,1] |\ \phi(z) \leqslant y\}) = \tilde{\mu}([0,x]) = \phi(x) = y = \lambda([0,y]).$$

Since $y \in [0,1]$ is arbitrary, we derive that $\phi_*\tilde{\mu} = \lambda$.

Now, let Z be the set of $y \in [0,1]$ such that $\phi^{-1}(\{y\})$ is a non-degenerate interval. Let $W = \phi^{-1}(Z)$. Then Z is countable, thus $\tilde{\mu}(W) = \lambda(Z) = 0$. The restriction $\psi := \phi_{|[0,1]\setminus W} : [0,1] \setminus W \to [0,1] \setminus Z$ is a Borel isomorphism such that $\psi_*\tilde{\mu} = \lambda$ (to see that ψ^{-1} is Borel, note that for every closed set $F \subset [0,1]$ we have $\psi(F \setminus W) = \phi(F) \setminus Z$ is Borel).

Finally, let $X_0 = X \setminus W$, $Y_0 = \psi(X_0)$, and $\theta := \psi_{|X_0} : X_0 \to Y_0$. Then θ satisfies the conclusion. ■

Corollary 3.10. *If Γ is a countable infinite abelian group, then $L(\Gamma)$ is $*$-isomorphic to $L^\infty([0,1],\lambda)$.*

Proof. Let μ denote the Haar measure of $\hat{\Gamma}$. Since Γ is infinite, $\hat{\Gamma}$ is infinite (e.g. because the Hilbert space $L^2(\hat{\Gamma},\mu)$ is isomorphic to $\ell^2(\Gamma)$ and therefore is infinite dimensional). Since $\mu(\{h\}) = \mu(\{e\})$, for every $h \in \hat{\Gamma}$, and $\mu(\hat{\Gamma}) = 1$, we deduce that μ has no atoms. The conclusion follows by combining Proposition 3.3 and Theorem 3.7. ■

Exercise 3.11. Let Γ be a finite abelian group and put $n = |\Gamma|$. Prove that $L(\Gamma)$ is $*$-isomorphic to $\ell^\infty(\{1,2,...,n\})$.

4. Abelian C*-algebras

4.1. **The spectrum and the spectral radius.** We start this section by establishing that the spectrum of a bounded operator is non-empty and a formula for the spectral radius.

Definition 4.1. A **Banach algebra** is an algebra A over $\mathbb{C}$ endowed with a norm $\|.\|$ such that $(A, \|.\|)$ is a Banach space and $\|xy\| \leqslant \|x\|\,\|y\|$, for all $x, y \in A$.

Lemma 4.2. *Let A be a unital Banach algebra. If $a \in A$ satisfies $\|a\| < 1$, then $1 - a$ is invertible, $(1-a)^{-1} = \sum_{k=0}^\infty a^k$ and $\|(1-a)^{-1}\| \leqslant \frac{1}{1-\|a\|}$. Moreover, if $\lambda \in \mathbb{C}$ and $|\lambda| > \|a\|$, then $\lambda \cdot 1 - a$ is invertible and $(\lambda\cdot 1 - a)^{-1} = \sum_{k=0}^\infty \lambda^{-n-1}a^n$.*

Proof. Let $x_n = \sum_{k=0}^n a^k$. For $m > n$, we have $\|x_m - x_n\| \leqslant \sum_{k=n+1}^m \|a\|^k \leqslant \frac{\|a\|^{n+1}}{1-\|a\|}$. Since $\|a\| < 1$, we get that $\{x_n\}$ is a Cauchy sequence. Let $x = \lim_{n\to\infty} x_n$. Then $\|x\| = \lim_{n\to\infty} \|x_n\| \leqslant \frac{1}{1-\|a\|}$. Since $(1-a)x_n = \sum_{k=0}^n (a^k - a^{k+1}) = 1 - a^{n+1}$, we get that $\|(1-a)x_n - 1\| \leqslant \|a\|^{n+1}$, for all n. It follows that $(1-a)x = 1$, which finishes the proof of the first assertion.

Since $\lambda\cdot 1 - a = \lambda(1 - \lambda^{-1}a)$, the moreover assertion is immediate. ■

Corollary 4.3. *Let A be a unital Banach algebra. The set G of invertible elements of A is open and the map $G \ni a \mapsto a^{-1} \in G$ is continuous.*

Proof. Let $a \in G$ and $b \in A$ such that $\|b-a\| \leqslant \frac{1}{2\|a^{-1}\|}$. Then $\|1-a^{-1}b\| \leqslant \|a^{-1}\|\,\|a-b\| \leqslant \frac{1}{2}$. Lemma 4.2 gives that $b^{-1}a = (a^{-1}b)^{-1}$ exists and $\|b^{-1}a\| \leqslant \frac{1}{1-\frac{1}{2}} = 2$. Thus, we derive that b^{-1} exists and $\|b^{-1}\| \leqslant \|b^{-1}a\|\,\|a^{-1}\| \leqslant 2\|a^{-1}\|$. Finally, we have that $b^{-1}-a^{-1} = b^{-1}(a-b)a^{-1}$ and therefore $\|b^{-1}-a^{-1}\| \leqslant \|b^{-1}\|\,\|a-b\|\,\|a^{-1}\| \leqslant \|a\|^{-2}\,\|a-b\|$. This proves the conclusion. ∎

Definition 4.4. Let A be a unital Banach algebra.

- The **spectrum** of $a \in A$ is given by $\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ not invertible}\}$.
- The **spectral radius** of $a \in A$ is defined as $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$.

Theorem 4.5. *$\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$.*

Proof. By Corollary 4.3, $\sigma(a) \subset \{\lambda \in \mathbb{C} |\ |\lambda| \leqslant \|a\|\}$ and $\mathbb{C} \setminus \sigma(a)$ is an open set. To show that $\sigma(a) \neq \emptyset$, we define the **resolvent** $R : \mathbb{C} \setminus \sigma(a) \to A$ by letting $R(\lambda) = (\lambda \cdot 1 - a)^{-1}$.

Let $\lambda_0 \in \mathbb{C} \setminus \sigma(a)$. Then $R(\lambda) - R(\lambda_0) = -(\lambda - \lambda_0)R(\lambda)R(\lambda_0)$. Since R is continuous on $\mathbb{R} \setminus \sigma(a)$, we get that $\lim\limits_{\lambda \to \lambda_0} \frac{R(\lambda)-R(\lambda_0)}{\lambda - \lambda_0} = -R(\lambda_0)^2$. This shows that R is an analytic function, in the sense that $\phi \circ R : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ is analytic, for every $\phi \in A^*$.

Now, by continuity of the inverse we get that $\lim\limits_{\lambda \to \infty} \|R(\lambda)\| = \lim\limits_{\lambda \to \infty} |\lambda|^{-1}\|(1-\lambda^{-1}a)^{-1}\| = 0$. So, if $\sigma(a) = \emptyset$, then $\phi \circ R : \mathbb{C} \to \mathbb{C}$ is a bounded analytic function, for every $\phi \in A^*$. By Liouville's theorem we get that $\phi \circ R$ is constant, hence $\phi(R(\lambda)) = 0$, for all $\lambda \in \mathbb{C}$. Since this holds for every $\phi \in A^*$, the Hahn-Banach theorem implies that $R(\lambda) = 0$, for all $\lambda \in \mathbb{C}$. This is a contradiction. ∎

Theorem 4.6. $r(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.

Proof. The conclusion will follow from two inequalities. First, if $\lambda \cdot 1 - a$ is not invertible, thus $\lambda^n \cdot 1 - a^n$ is not invertible, hence $|\lambda^n| \leqslant \|a^n\|$, for all n. This proves that $r(a) \leqslant \liminf\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.

On the other hand, recall that $R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1}a^n$, whenever $|\lambda| > \|a\|$. Since R is analytic on $\{\lambda \in \mathbb{C} |\ |\lambda| \geqslant r(a)\}$, the last series converges for $|\lambda| > r(a)$. In particular, $\lim\limits_{n\to\infty} \lambda^{-n-1}a^n = 0$ and therefore $|\lambda| \geqslant \limsup\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$, whenever $|\lambda| > r(a)$. This shows that $r(a) \geqslant \limsup\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. ∎

Theorem 4.7 (Gelfand-Mazur)**.** *If A is a unital Banach algebra in which every non-zero element is invertible, then $A \cong \mathbb{C} \cdot 1$.*

Proof. Assume that there is an element $a \in A \setminus \mathbb{C} \cdot 1$. Then $\lambda \cdot 1 - a$ is invertible, for every $\lambda \in \mathbb{C}$. Thus, $\sigma(a)$ is empty, which contradicts Theorem 4.5. ∎

Definition 4.8. Let A be a unital abelian Banach algebra. We let $\Sigma(A)$ be the set of all non-zero homomorphisms $\varphi : A \to \mathbb{C}$, and call it the **maximal ideal space** (or **dual** or **spectrum**) of A.

Corollary 4.9. *If $I \subset A$ is maximal ideal, then there is $\varphi \in \Sigma(A)$ such that $I = \ker(\varphi)$.*

Proof. Recall that an **ideal** $I \subset A$ is a vector subspace such that $ax \in I$, for all $x \in I$ and $a \in A$.

Let us first show that I is closed. Since I is a proper ideal, we get that $\|1-x\| \leqslant 1$, for all $x \in I$ (otherwise, x would be invertible by Lemma 4.2). This implies that $1 \notin \bar{I}$, hence $\bar{I}$ is a proper ideal of A. Since $I \subset \bar{I}$ and I is a maximal ideal, we get that $I = \bar{I}$.

Since I is a closed ideal, the quotient A/I is a Banach algebra, where $\|a+I\| = \inf_{x\in I}\|a+x\|$. Moreover, if $a \in A \setminus I$, then $aA + I$ is an ideal of A which contains I. Hence, $Aa + I = A$ and so we can find $x \in A, y \in I$ such that $ax + y = 1$. Thus, $(a+I)(x+I) = 1 + I$ and therefore $a + I$ is invertible in A/I. This shows that every non-zero element in A/I is invertible. Theorem 4.7 implies that $A/I = \mathbb{C}\cdot 1$. Finally, the quotient homomorphism $\varphi : A \to A/I$ satisfies $\ker(\varphi) = I$. ■

4.2. **Abstract abelian C*-algebras.** Recall that we defined a concrete C*-algebra to be a norm closed $*$-subalgebra $A \subset \mathbb{B}(H)$.

Definition 4.10. An **abstract C*-algebra** is a Banach algebra $(A, \|.\|)$ together with an **adjoint** operation $* : A \to A$ such that

$$(a+b)^* = a^* + b^*,\ (\lambda a)^* = \bar{\lambda}a^*,\ (a^*)^* = a,\ (ab)^* = b^*a^*, \quad \text{for all } a, b \in A \text{ and } \lambda \in \mathbb{C}, \text{ and}$$

$$\|a^*a\| = \|a\|^2, \quad \text{for all } a \in A.$$

Remark 4.11. By Exercise 15.3, any concrete C*-algebra is an abstract C*-algebra. As we will note in the next section, the converse is also true.

If X be a compact Hausdorff space, then $C(X)$ is an abstract C*-algebra, where the norm and adjoint given by $\|f\|_\infty = \sup_{x\in X}|f(x)|$ and $f^*(x) = \overline{f(x)}$. The goal of this subsection is to prove that every abstract abelian C*-algebra arises this way.

Lemma 4.12. *Let A be a unital abelian C*-algebra. If $\varphi \in \Sigma(A)$, then $\|\varphi\| = 1$ and $\varphi(a^*) = \overline{\varphi(a)}$, for all $a \in A$.*

Proof. Since $\varphi(1) = 1$, we have that $\|\varphi\| \geqslant 1$. We claim that $\|\varphi\| \leqslant 1$. If $\|\varphi\| > 1$, then we can find $a \in A$ such that $\|a\| < 1$ and $|\varphi(a)| = 1$. Let $b = \sum_{n=1}^\infty a^n$. Then $a + ab = b$ and applying φ gives that $1 + \varphi(b) = \varphi(b)$, which is absurd.

Next, let $a \in A$ be self-adjoint. Then for every $t \in \mathbb{R}$ we have that

$$|\varphi(a+it)|^2 \leqslant \|a+it\|^2 = \|(a+it)(a-it)\| = \|a^2+t^2\| \leqslant \|a^2| + t^2.$$

If we write $\varphi(a) = x + iy$, then $|\varphi(a+it)|^2 = x^2 + (t+y)^2$. The above inequality rewrites as $x^2 + 2ty \leqslant \|a\|^2$, for all $t \in \mathbb{R}$. This forces $y = 0$ and thus $\varphi(a) \in \mathbb{R}$.

Finally, let $a \in A$ be arbitrary and write $a = b + ic$, where $b, c \in A$ are self-adjoint. Then we have that $\varphi(a^*) = \varphi(b^*) - i\varphi(c^*) = \varphi(b) - i\varphi(c) = \overline{\varphi(a)}$. ■

Lemma 4.13. *By Lemma 4.12 we have that $\Sigma(A) \subset \{\varphi \in A^* |\ \|\varphi\| = 1\}$. Then $\Sigma(A)$ is a compact Hausdorff space with respect to the relative weak* topology.*

Proof. If $\varphi_i : A \to \mathbb{C}$ is a net of homomorphisms such that $\varphi_i(a) \to \varphi(a)$, for every $a \in A$, then φ is a homomorphism. This shows that $\Sigma(A)$ is weak*-closed. Since $\{\varphi \in A^* |\ \|\varphi\| = 1\}$ is weak*-compact by Alaoglu's theorem, the conclusion follows. ■

Theorem 4.14. *Assume that A is a unital abelian C*-algebra. Then the* **Gelfand transform** *$\Gamma : A \to C(\Sigma(A))$ given by $\Gamma(a)(\varphi) = \varphi(a)$ is a $*$-isomorphism.*

Proof. Firslty, since $\Gamma(a^*)(\varphi) = \varphi(a^*) = \overline{\varphi(a)} = \overline{\Gamma(a)(\varphi)} = \overline{\Gamma(a)}(\varphi)$, Γ is a $*$-homomorphism.

Secondly, let $a \in A$ be self adjoint. Then $\|\Gamma(a)\| = \sup_{\varphi\in\Sigma(A)}|\varphi(a)| \leqslant \|a\|$. Since a is self-adjoint, $r(a) = \|a\|$. If $\lambda \in \sigma(a)$, then $\lambda \cdot 1 - a$ is not invertible and is therefore contained in a maximal ideal of A. By Lemma 4.9 we can find $\varphi \in \Sigma(A)$ such that $\varphi(\lambda\cdot 1 - a) = 0$. Thus, $\lambda = \varphi(a)$. It follows that $\|\Gamma(a)\| = \sup_{\varphi\in\Sigma(A)}|\varphi(a)| \geqslant r(a) = \|a\|$. Altogether, we get that $\|\pi(a)\| = \|a\|$, whenever $a \in A$ is self-adjoint. Thus, $\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\| = \|\pi(a^*a)\| = \|a^*a\| = \|a\|^2$ and hence $\|\pi(a)\| = \|a\|$, for every $a \in A$. We have proven that Γ is an isometry.

Finally, since π is an isometric $*$-homomorphism, $\Gamma(A)$ is a closed subalgebra of $C(\Sigma(A))$ which contains 1 and is closed under complex conjugation. Since $\Gamma(A)$ clearly separates points in $\Sigma(A)$, the Stone-Weierstrass theorem implies that $\Gamma(A) = C(\Sigma(A))$. ■

5. Continuous functional calculus and applications

5.1. Continuous functional calculus.

Theorem 5.1. *Let $a \in A$ be a normal element. Denote by $C^*(a)$ the C*-algebra generated by a. Then there exists a $*$-isomorphism $\pi : C^*(a) \to C(\sigma(a))$ such that $\pi(a) = z$.*

Proof. Since a is normal, $A := C^*(a)$ is an abelian C*-algebra. By Theorem 4.14, the Gelfand transform $\Gamma : A \to C(\Sigma(A))$ is a $*$-isomorphism.

Let $\rho : \Sigma(A) \to \sigma(a)$ be given by $\rho(\varphi) = \varphi(a)$. Note that ρ is well defined since $\varphi(\varphi(a)\cdot 1 - a) = 0$ and thus $\varphi(a)\cdot 1 - a$ is not invertible. The proof of Theorem 4.14 shows that ρ is also surjective. Moreover, ρ is injective. If $\varphi_1(a) = \varphi_2(a)$, then $\varphi_1(a^*) = \varphi_2(a^*)$ and therefore $\varphi_1(P(a, a^*)) = \varphi_2(P(a, a^*))$, for every polynomial P in a and a^*. This implies that $\varphi_1 \equiv \varphi_2$. Since ρ is continuous and $\Sigma(A)$ is compact, we conclude that ρ is a homeomorphism.

Then $\tau : C(\Sigma(A)) \to C(\sigma(a))$ given by $\tau(f) = f \circ \rho^{-1}$ is a $*$-isomorphism. Finally, we define $\pi := \tau \circ \Gamma : A \to C(\sigma(a))$. Then π is a $*$-isomorphism and $\pi(a) = z$. ■

Corollary 5.2 (continuous functional calculus)**.** *In the notation from Theorem 5.1, we define $f(a) := \pi^{-1}(f)$, for every $f \in C(\sigma(a))$. Then $\|f(a)\| = \|f\|_\infty$ and $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Since the Gelfand transform Γ is isometric, π is isometric. This implies the first assertion. The second assertion is immediate since $\sigma(f(a)) = \sigma(f) = f(\sigma(a))$. ■

In the rest of this section, we derive several consequences of continuous functional calculus.

Definition 5.3. An element a of a C*-algebra A is called **positive** if $a = a^*$ and $\sigma(a) \subset [0, \infty)$.

Exercise 5.4. Prove that an operator $T \in \mathbb{B}(H)$ is positive if and only if $\langle T\xi, \xi\rangle \geqslant 0$, for all $\xi \in H$.

Corollary 5.5. *Let A be a C*-algebra and $a \in A$.*

1. *If a is self-adjoint, then $\sigma(a) \subset \mathbb{R}$, $\|a\| - a$ is positive, and there exist unique positive elements $b, c \in A$ such that $a = b - c$ and $bc = cb = 0$.*
2. *If a is a unitary, then $\sigma(a) \subset \mathbb{T}$.*
3. *If a is positive, then there exists a unique positive element $b \in A$ such that $a = b^2$.*

Proof. (1) Let $A = C^*(a)$. If $\varphi \in \Sigma(A)$, then $\varphi(a) = \varphi(a^*) = \overline{\varphi(a)}$, hence $\varphi(a) \in \mathbb{R}$. Since $\sigma(a) \subset \{\varphi(a) | \varphi \in \Sigma(A)\}$, we get that $\sigma(a) \subset \mathbb{R}$.

Since $\sigma(a) \subset \mathbb{R}$, we get that $\sigma(a) \subset [-\|a\|, \|a\|]$. Thus, $\sigma(\|a\| - a) = \|a\| - \sigma(a) \subset [0, 2\|a\|]$, and since $\|a\| - a$ is self-adjoint, we deduce that it is positive.

Let $f, g \in C(\sigma(a))$ be given by $f(t) = \max\{t, 0\}$ and $g(t) = -\min\{t, 0\}$. Put $b = f(a)$ and $c = g(a)$. Since $f, g \geqslant 0$, Corollary 5.5 implies that $b, c \geqslant 0$. Since $f(t) - g(t) = t$ and $fg = 0$, we also get that $b - c = a$ and $bc = cb = 0$. This proves the existence assertion. For the uniqueness part, just note that the C*-algebra generated by any such b, c is abelian, hence isomorphic to $C(X)$, for some compact Hausdorff space X.

(2) If $\varphi \in \Sigma(A)$, then $|\varphi(a)|^2 = \overline{\varphi(a)}\varphi(a) = \varphi(a^*)\varphi(a) = \varphi(a^*a) = 1$, hence $|\varphi(a)| = 1$. Since $\sigma(a) \subset \{\varphi(a)|\varphi \in \Sigma(A)\}$, we get that $\sigma(a) \subset \mathbb{T}$.

(3) Since $\sigma(a) \subset [0, \infty)$ we can define $h \in C(\sigma(a))$ by letting $h(t) = \sqrt{t}$. Then $b = h(a)$ is positive and $b^2 = a$. The uniqueness assertion follows as sketched in the proof of (1). ∎

Exercise 5.6. If A is a C*-algebra, prove that every element $a \in A$ can be written as a linear combination of four unitary elements from A.

5.2. **The Gelfand-Naimark-Segal construction.**

Theorem 5.7 (Gelfand-Naimark-Segal)**.** *Every abstract C*-algebra is isometrically $*$-isomorphic to a concrete C*-algebra.*

For the proof of Theorem 5.7, see [Co99, Theorem 7.10]. Here, we will only emphasize the main tool in the proof of Theorem 5.7, the so-called **Gelfand-Naimark-Segal** (**GNS**) **construction**.

Definition 5.8. Let A be a C*-algebra. A linear functional $\varphi : A \to \mathbb{C}$ is called **positive** if $\varphi(a^*a) \geq 0$, for all $a \in A$. If A is unital, then a positive linear functional $\varphi : A \to \mathbb{C}$ is called a **state** if $\varphi(1) = 1$. A positive linear functional $\varphi : A \to \mathbb{C}$ is called **faifthul** if $\varphi(a^*a) = 0 \Rightarrow a = 0$.

Exercise 5.9. (the Cauchy-Schwarz inequality) Let A be a C*-algebra and $\varphi : A \to \mathbb{C}$ be a positive linear functional. Prove that $|\varphi(y^*x)|^2 \leqslant \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.

Exercise 5.10. Let A be a unital C*-algebra and $\varphi : A \to \mathbb{C}$ be a positive linear functional. Prove that φ is bounded and $\|\varphi\| = \varphi(1)$.

Theorem 5.11 (the GNS construction)**.** *Let A be a unital C*-algebra and $\varphi : A \to \mathbb{C}$ be a state. Then there exist a Hilbert space H_φ, a $*$-homomorphism $\pi_\varphi : A \to \mathbb{B}(H_\varphi)$ and a unit vector $\xi_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle$, for all $a \in A$.*

Proof. Let $I = \{x \in A \mid \varphi(x^*x) = 0\}$. Then I is a closed left ideal of A. Indeed, I is closed since φ is bounded. Moreover, if $a \in A$ and $x \in I$, then $\|a^*a\| - a^*a$ is positive by Corollary 5.5 (1) and thus can be written as $\|a^*a\| - a^*a = b^*b$, for some $b \in A$, by Corollary 5.5 (3). Since φ is positive, we get that $\varphi(x^*b^*bx) \geq 0$ and therefore $0 \leq \varphi(x^*a^*ax) \leq \|a^*a\|\varphi(x^*x) = 0$, implying that $ax \in I$.

Consider the vector space A/I and for $x, y \in A$, define $\langle x + I, y + I\rangle = \varphi(y^*x)$. Then $\langle \cdot, \cdot\rangle$ defines an inner product on A/I. Let H_φ to be the completion of A/I w.r.t. the norm defined by this inner product. If $a, x \in A$, then by using the inequality proved above, we have

$$\|ax + I\|^2 = \langle ax + I, ax + I\rangle = \varphi(x^*a^*ax) \leq \|a\|^2\varphi(x^*x) = \|a\|^2\|x + I\|^2.$$

Thus, if $a \in A$, then the map $\pi_\varphi : A/I \to A/I$ defined by $\pi_\varphi(a)(x + I) = ax + I$ extends to a bounded operator on H_φ satisfying $\|\pi_\varphi(a)\| \leq \|a\|$. It is easy to see that $\pi_\varphi : A \to \mathbb{B}(H_\varphi)$ is a $*$-homomorphism, $\xi_\varphi = 1 + I \in H_\varphi$ is a unit vector and $\langle \pi_\varphi(\xi_\varphi), \xi_\varphi\rangle = \varphi(a)$, for all $a \in A$. ∎

Remark 5.12. Let X be a compact Hausdorff space. Then $C(X)$ is an abstract C*-algebra, where the norm and adjoint given by $\|f\|_\infty = \sup_{x\in X} |f(x)|$ and $f^*(x) = \overline{f(x)}$. Let $\mathcal{M}(X)$ denote the set of regular Borel probability measures on X. By Riesz's representation theorem, $\mathcal{M}(X)$ is equal to the positive part of the unital ball of $C(X)^*$. Let $H = \oplus_{\mu\in\mathcal{M}(X)} L^2(X, \mu)$. The $*$-homomorphism $\pi : C(X) \to \mathbb{B}(H)$ given by multiplication, is isometric. This a concrete representation of $C(X)$.

For a general abstract C*-algebra A, one considers the set $\mathcal{S}(A)$ of all states $\varphi : A \to \mathbb{C}$, and shows that the $*$-homomorphism $\pi = \oplus_{\varphi\in\mathcal{S}(A)}\pi_\varphi : A \to \mathbb{B}(\oplus_{\varphi\in\mathcal{S}(A)} H_\varphi)$ is isometric.

5.3. **Continuity of $*$-homomorphisms.**

Lemma 5.13. *Let $\pi : A \to B$ be a $*$-homomorphism between two abstract C^*-algebras. Then π is contractive: $\|\pi(a)\| \leqslant \|a\|$, for all $a \in A$.*

Proof. Let $a \in A$ be a self-adjoint element. Then $\|a^2\| = \|a^*a\| = \|a\|^2$. Since a^{2^n} is self-adjoint, we get that $\|a^{2^{n+1}}\| = \|a^{2^n}\|^2$ and by induction it follows that $\|a^{2^n}\| = \|a\|^{2^n}$, for all $n \geqslant 1$. As a consequence $r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|$. (If a is only assumed normal, then for all $n \geqslant 1$ we have $\|a^{2^n}\|^2 = \|a^{2^n*}a^{2^n}\| = \|(a^*a)^{2^n}\| = \|a\|^{2^{n+1}}$. We get that $r(a) = \|a\|$ in this case as well)

Let $a \in A$. If $\lambda \cdot 1 - a^*a$ invertible, then $\lambda \cdot 1 - \pi(a^*a)$ is invertible, thus $\sigma(\pi(a^*a)) \subset \sigma(a^*a)$. This fact implies that $\|\pi(a)\|^2 = \|\pi(a^*a)\| = r(\pi(a^*a)) \leqslant r(a^*a) = \|a^*a\| = \|a\|^2$, hence π is contractive. ■

Corollary 5.14. *Any injective $*$-homomorphism $\pi : A \to B$ between two C^*-algebras is an isometry.*

Proof. Let $a \in A$ be a self-adjoint element. Consider the inclusion $i : \sigma(\pi(a)) \to \sigma(a)$ and the $*$-homomorphism $\rho : C(\sigma((a)) \to C(\sigma(\pi(a)))$ given by $\rho(f) = f \circ i$. Let $\Gamma_1 : C^*(a) \to C(\sigma(a))$ and $\Gamma_2 : C^*(\pi(a)) \to C(\sigma(\pi(a)))$ be the Gelfand transforms. Then $\rho(x) = \Gamma_2^{-1} \circ \pi \circ \Gamma_1(x)$, for every $x \in C^*(a)$. Since π is injective, we get that ρ is injective. This implies that $\sigma(\pi(a)) = \sigma(a)$, hence $\|\pi(a)\| = r(\pi(a)) = r(a) = \|a\|$. It is now clear that π is an isometry. ■

5.4. **Kaplansky's density theorem.**

Theorem 5.15 (Kaplansky's density theorem)**.** *Let $M \subset \mathbb{B}(H)$ be a unital von Neumann algebra. Let $A \subset M$ be a C^*-subalgebra such that $\overline{A}^{SOT} = M$. Then the following hold:*

(1) $\overline{A_1}^{SOT} = M_1$, where $A_1 = \{a \in A |\ \|a\| \leqslant 1\}$.
(2) $\overline{A_{1,sa}}^{SOT} = M_{1,sa}$, where $A_{1,sa} = \{a \in A |\ \|a\| \leqslant 1$ and $a^* = a\}$.
(3) $\overline{A_{sa}} = M_{sa}$, where $A_{sa} = \{a \in A |\ a^* = a\}$.

Proof. (3) Let $x \in M_{sa}$ and $x_i \in A$ such that $x_i \to x$ (SOT). Then $x_i^* \to x^* = x$ (WOT) and therefore $\frac{1}{2}(x_i + x_i^*) \to x$ (WOT). By using Lemma 1.22 we derive that $x \in \overline{A_{sa}}^{WOT} = \overline{A_{sa}}^{SOT}$.

(2) Let $f : \mathbb{R} \to [-1, 1]$ be given by $f(t) = \frac{2t}{1+t^2}$. Then $g = f_{|[-1,1]}$ is a homeomorphism of $[-1, 1]$. Let $g = (f_{|[-1,1]})^{-1} : [-1, 1] \to [-1, 1]$.

Let $x \in M_{1,sa}$ and put $y = g(x) \in M_{1,sa}$. Then $f(y) = x$. By part (1), there exists a net $y_i \in A_{sa}$ such that $y_i \to y$ (SOT). We claim that $f(y_i) \to f(y) = x$ (SOT). Since $x_i = f(y_i) \in A_{1,sa}$ (by continuous functional calculus) this claim implies that $x \in \overline{A_{1,\mathrm{sa}}}^{SOT}$, as desired.

To prove the claim, note that

$$f(y_i) - f(y) = 2y_i(1 + y_i)^{-1} - 2y(1 + y^2)^{-1} = (1 + y_i)^{-1}(2y_i(1 + y^2) - 2(1 + y_i^2))(1 + y^2)^{-1} =$$
$$(1 + y_i^2)^{-1}(2(y_i - y) + 2y_i(y - y_i)y)(1 + y^2)^{-1}.$$

Let $\xi \in H$ and put $\eta = (1 + y^2)^{-1}(\xi)$. Then

$$\|(f(y_i) - f(y))(\xi)\| \leqslant 2\|(y_i - y)(\eta)\| + 2\|y_i(1 + y_i^2)^{-1}\|\ \|(y_i - y)(y\eta)\|$$

which implies that $\|(f(y_i) - f(y))(\xi)\| \to 0$, as claimed.

(1) We use a "matrix trick". Consider the inclusions $\mathbb{M}_2(A) \subset \mathbb{M}_2(M) \subset \mathbb{M}_2(\mathbb{B}(H)) = \mathbb{B}(H^2)$. Let $x \in M_1$ and define $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$. Then $\tilde{x} \in \mathbb{M}_2(M)_{1,sa}$. Since $\overline{\mathbb{M}_2(A)}^{SOT} = \mathbb{M}_2(\overline{A}^{SOT}) = \mathbb{M}_2(M)$, by applying part (2) we can find a net $y_i \in \mathbb{M}_2(A)_{1,sa}$ such that $y_i \to \tilde{x}$ (SOT).

Write $y_i = \begin{pmatrix} y^i_{11} & y^i_{12} \\ y^i_{21} & y^i_{22} \end{pmatrix}$, where $y^i_{p,q} \in A$. Let $\xi \in H$. Since $\begin{pmatrix} y^i_{12}(\xi) \\ y^i_{22}(\xi) \end{pmatrix} = y_i \begin{pmatrix} 0 \\ \xi \end{pmatrix} \to \tilde{x} \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} x(\xi) \\ 0 \end{pmatrix}$, we get that $y^i_{12} \to x$ (SOT). Since $\|y^i_{12}(\xi)\| \leqslant \|y_i \begin{pmatrix} 0 \\ \xi \end{pmatrix}\| \leqslant \|y_i\| \, \|\xi\| \leqslant \|\xi\|$, we get that $\|y^i_{12}\| \leqslant 1$. Therefore, $x \in \overline{A_1}^{SOT}$, as claimed. ∎

6. The spectral theorem

Let $A \subset \mathbb{B}(H)$ be a concrete abelian C*-algebra (e.g. take A to be the C*-algebra generated by a normal operator). Then A is "abstractly" $*$-isomorphic to $C(\Sigma)$, where Σ is the maximal ideal space of A. This result, although very useful, does not explain how A acts on H. In this section we prove the so-called spectral theorem which gives a description of all $*$-representations of $C(\Sigma)$.

Remark 6.1. Assume $\Sigma = \{x_1, ..., x_n\}$ is finite (equivalently, $C(\Sigma)$ is finite dimensional) and let $\pi : C(\Sigma) \to \mathbb{B}(H)$ be a $*$-homomorphism. Then $e_i = \pi(1_{\{x_i\}})$, $i = 1, ..., n$, are commuting projections. For all $f \in C(\Sigma)$, we have that $\pi(f) = \pi(\sum_{i=1}^n f(x_i) 1_{\{x_i\}}) = \sum_{i=1}^n f(x_i) e_i =$ "$\int_\Sigma f$ de".

Notation. Let Σ be a compact Hausdorff space. We denote by Ω the σ-algebra of Borel subsets of Σ and by $B(\Sigma)$ the algebra of bounded Borel functions $f : \Sigma \to \mathbb{C}$.

Theorem 6.2 (the spectral theorem). *Let $\pi : C(\Sigma) \to \mathbb{B}(H)$ be a $*$-representation. Then there exists a unique regular spectral measure $E : \Omega \to \mathbb{B}(H)$ such that*

$$\pi(f) = \int_\Sigma f \, dE, \quad \textit{for every } f \in C(\Sigma).$$

Definition 6.3. A projection-valued **spectral measure** for (Σ, Ω) is a map $E : \Omega \to \mathbb{B}(H)$ that satisfies the following conditions:

(1) $E(\Delta)$ is a projection, for every $\Delta \in \Omega$.
(2) $E(\emptyset) = 0$ and $E(\Sigma) = 1$.
(3) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$, for every $\Delta_1, \Delta_2 \in \Omega$.
(4) $E(\cup_{n=1}^\infty \Delta_n) = \sum_{n=1}^\infty E(\Delta_n)$, in the SOT, for any pairwise disjoint sets $\{\Delta_n\}_{n=1}^\infty$ from Ω.

Example 6.4. Let μ be a Borel probability measure on Σ. Let $\pi : L^\infty(\Sigma, \mu) \to \mathbb{B}(L^2(\Sigma, \mu))$ be the $*$-homomorphism given by $\pi_f(\xi) = f\xi$. Then $E(\Delta) = \pi_{1_\Delta}$ defines a spectral measure.

Lemma 6.5. *Let $E : \Omega \to \mathbb{B}(H)$ be a spectral measure. If $\xi, \eta \in H$, then $E_{\xi,\eta}(\Delta) = \langle E(\Delta)(\xi), \eta \rangle$ defines a complex-valued measure on Ω with $\|E_{\xi,\eta}\| \leqslant \|\xi\| \, \|\eta\|$.*

Proof. It is clear that $E_{\xi,\eta}$ is a complex-valued measure on Ω. To prove the total variation assertion, let $\Delta_1, ..., \Delta_n$ be pairwise disjoint sets. Let $\alpha_i \in \mathbb{T}$ such that $|E_{\xi,\eta}(\Delta_i)| = \alpha_i E_{\xi,\eta}(\Delta_i)$. Then we have that $\sum_{i=1}^n |E_{\xi,\eta}(\Delta_i)| = \langle \sum_{i=1}^n \alpha_i E(\Delta_i)\xi, \eta \rangle \leqslant \| \sum_{i=1}^n \alpha_i E(\Delta_i)\xi\| \, \|\eta\|$. Since we also have that

$$\| \sum_{i=1}^n \alpha_i E(\Delta_i)\xi\|^2 = \sum_{i=1}^n \langle E(\Delta_i)\xi, \xi \rangle = \langle E(\cup_{i=1}^n \Delta_i)\xi, \xi \rangle \leqslant \|\xi\|^2,$$

we conclude that $\sum_{i=1}^n |E_{\xi,\eta}(\Delta_i)| \leqslant \|\xi\| \, \|\eta\|$, as desired. ∎

Definition 6.6. A spectral measure E is called **regular** if $E_{\xi,\eta}$ is regular, for every $\xi, \eta \in H$.

Lemma 6.7. *Let $E : \Omega \to \mathbb{B}(H)$ be a spectral measure. Then for every $f \in B(\Sigma)$, there exists an operator $\pi(f) \in \mathbb{B}(H)$ such that $\|\pi(f)\| \leqslant \|f\|_\infty$ and $\langle \pi(f)\xi, \eta\rangle = \int_\Sigma f \; dE_{\xi,\eta}$, for all $\xi, \eta \in H$. Moreover, the map $\pi : B(\Sigma) \to \mathbb{B}(H)$ is a $*$-homomorphism.*

Proof. Firstly, let $f \in B(\Sigma)$. Since the map $H \times H \ni (\xi, \eta) \to \int_\Sigma f \; \mathrm{d}E_{\xi,\eta}$ is sesquilinear and satisfies $|\int_\Sigma f \; \mathrm{d}E_{\xi,\eta}| \leqslant \|f\|_\infty \|E_{\xi,\eta}\| \leqslant \|f\|_\infty \|\xi\|\|\eta\|$, the existence of $\pi(f)$ is a consequence of Riesz's representation theorem.

Secondly, let $\Delta \in \Omega$. Then $\langle \pi(1_\Delta)\xi, \eta\rangle = \int_\Sigma 1_\Delta \mathrm{d}E_{\xi,\eta} = E_{\xi,\eta}(\Delta) = \langle E(X)\xi, \eta\rangle$ and therefore $\pi(1_\Delta) = E(\Delta)$. We get that $\pi(1_{\Delta_1 \cap \Delta_2}) = \pi(1_{\Delta_1})\pi(1_{\Delta_2})$, for every $\Delta_1, \Delta_2 \in \Omega$, and further that $\pi(f_1 f_2) = \pi(f_1)\pi(f_2)$, for any simple functions $f_1, f_2 \in B(X)$. Since $\|\pi(f)\| \leqslant \|f\|_\infty$, for every $f \in B(\Sigma)$, by approximating bounded Borel functions with simple functions, we conclude that π is multiplicative. It follows that π is a $*$-homomorphism. ■

Before proving the spectral theorem, we need one additional result.

Lemma 6.8. *Let $\pi : C(\Sigma) \to \mathbb{B}(H)$ be a $*$-homomorphism. Then there exists a $*$-homomorphism $\tilde{\pi} : B(\Sigma) \to \mathbb{B}(H)$ such that $\tilde{\pi}_{|C(\Sigma)} = \pi$.*

Moreover, if $f \in B(\Sigma)$ and $f_i \in B(\Sigma)$ is a net such that $\int_\Sigma f_i \; d\mu \to \int_\Sigma f \; d\mu$, for every regular Borel measure μ on Σ, then $\tilde{\pi}(f_i) \to \tilde{\pi}(f)$ in the WOT.

Proof. Let $\xi, \eta \in H$. Note that $C(\Sigma) \ni f \to \langle \pi(f)\xi, \eta\rangle \in \mathbb{C}$ is a linear functional such that $|\langle \pi(f)\xi, \eta\rangle| \leqslant \|\pi(f)\| \; \|\xi\| \; \|\eta\| \leqslant \|f\| \; \|\xi\| \; \|\eta\|$. Riesz's representation theorem implies that there exists a regular complex Borel measure $\mu_{\xi,\eta}$ on Σ such that $\int_\Sigma f \; \mathrm{d}\mu_{\xi,\eta} = \langle \pi(f)\xi, \eta\rangle$, for all $f \in C(\Sigma)$, and $\|\mu_{\xi,\eta}\| \leqslant \|\xi\| \; \|\eta\|$. Note that $\overline{\mu_{\xi,\eta}} = \mu_{\eta,\xi}$, so the map $(\xi, \eta) \to \mu_{\xi,\eta}$ is sesquilinear.

Next, let $f \in B(\Sigma)$. Repeating the argument from the proof of Lemma 6.7 shows that there exists an operator $\tilde{\pi}(f) \in \mathbb{B}(H)$ such that $\|\tilde{\pi}(f)\| \leqslant \|f\|_\infty$ and $\langle \tilde{\pi}(f)\xi, \eta\rangle = \int_\Sigma f \; \mathrm{d}\mu_{\xi,\eta}$, for all $\xi, \eta \in H$. It is clear that $\tilde{\pi}(f) = \pi(f)$, if $f \in C(\Sigma)$, so the last assertion is verified.

It is easy to see that $\tilde{\pi}$ is linear and $*$-preserving, so it remains to argue that $\tilde{\pi}$ is multiplicative. To this end, let $f \in B(\Sigma)$ and $g \in C(\Sigma)$. By Lemma 3.4, we can find a net $f_i \in C(\Sigma)$ such that $\|f_i\|_\infty \leqslant \|f\|_\infty$, for all i, and $\int_\Sigma f_i \; \mathrm{d}\mu \to \int_\Sigma f \; \mathrm{d}\mu$, for every regular Borel measure μ on Σ. Since $\mu_{\xi,\eta}$ is a regular Borel measure it follows that $\langle \pi(f_i)\xi, \eta\rangle = \int_\Sigma f_i \; \mathrm{d}\mu_{\xi,\eta} \to \int_\Sigma f \; \mathrm{d}\mu_{\xi,\eta} = \langle \tilde{\pi}(f)\xi, \eta\rangle$, for all $\xi, \eta \in H$. Thus, $\pi(f_i) \to \tilde{\pi}(f)$ in the WOT. Similarly, $\pi(f_i g) \to \tilde{\pi}(fg)$ in the WOT. Since $\pi(f_i g) = \pi(f_i)\pi(g)$, for all i, we deduce that $\tilde{\pi}(fg) = \tilde{\pi}(f)\pi(g)$, for all $f \in B(\Sigma)$ and $g \in C(\Sigma)$.

Finally, let $f, g \in B(\Sigma)$. By approximating g with continuous functions as above and using the last identity, it follows similarly that $\tilde{\pi}(fg) = \tilde{\pi}(f)\tilde{\pi}(g)$. Thus, $\tilde{\pi}$ is multiplicative.

For the moreover assertion, let $f, f_i \in B(\Sigma)$ as in the hypothesis. Then for every $\xi, \eta \in H$ we have that $\langle \tilde{\pi}(f_i)\xi, \eta\rangle = \int_\Sigma f_i \; \mathrm{d}\mu_{\xi,\eta} \to \int_\Sigma f \; \mathrm{d}\mu_{\xi,\eta} = \langle \tilde{\pi}(f)\xi, \eta\rangle$. Therefore, $\tilde{\pi}(f_i) \to \tilde{\pi}(f)$ in the WOT. ■

Proof of Theorem 6.2. By Lemma 6.8, π extends to a $*$-homomorphism $\tilde{\pi} : B(\Sigma) \to \mathbb{B}(H)$. We define $E : \Omega \to \mathbb{B}(H)$ by letting $E(\Delta) = \tilde{\pi}(1_\Delta)$. We claim that E is a spectral measure.

Firstly, since $1_\Delta^2 = \overline{1_\Delta} = 1_\Delta$, by applying $\tilde{\pi}$ we get that $E(\Delta)^2 = E(\Delta)^* = E(\Delta)$, hence $E(\Delta)$ is a projection. Secondly, since $\tilde{\pi}(0) = 0$ and $\tilde{\pi}(1) = 1$, we get that $E(\emptyset) = 0$ and $E(\Sigma) = 1$. Thirdly, if $\Delta_1, \Delta_2 \in \Omega$, then $1_{\Delta_1} 1_{\Delta_2} = 1_{\Delta_1 \cap \Delta_2}$ and applying $\tilde{\pi}$ yields that $E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2)$.

Finally, let $\{\Delta_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint Borel subsets of Σ. Put $X_k = \cup_{n=1}^k \Delta_n$ and $X = \cup_{n=1}^\infty \Delta_n$. Then $\lim\limits_{k\to\infty} \mu(X \setminus X_k) = 0$, for every Borel measure μ on Σ. The moreover part of Lemma 6.8 gives that $E(X \setminus X_k) = \tilde{\pi}(1_{X \setminus X_k}) \to 0$ in the WOT. If $\xi \in H$, then we have that $\|E(X \setminus X_k)\xi\|^2 = \langle E(X \setminus X_k)\xi, \xi\rangle \to 0$, which shows that $E(X \setminus X_k) \to 0$ in the SOT. Hence $E(X_k) \to E(X)$ in the SOT. Since $E(X_k) = \sum_{n=1}^k E(\Delta_n)$, the claim is proven.

If $\xi, \eta \in H$, then $E_{\xi,\eta}(\Delta) = \mu_{\xi,\eta}$. Since $\mu_{\xi,\eta}$ is a Borel regular measure by construction (see the proof of Lemma 6.8), we get that E is a regular. Thus, E is a regular spectral measure.

Further, by Lemma 6.7 $\rho : B(\Sigma) \to \mathbb{B}(H)$ given by $\rho(f) = \int_\Sigma f \, \mathrm{d}E$ is a $*$-homomorphism. Then $\rho(1_\Delta) = \int_\Sigma 1_\Delta \, \mathrm{d}E = E(\Delta) = \tilde{\pi}(1_\Delta)$, for every $\Delta \in \Omega$. It follows that $\rho(f) = \tilde{\pi}(f)$, for every simple function $f \in B(\Sigma)$. Since simple functions are dense in $B(\Sigma)$ and both $\rho, \tilde{\pi}$ are contractive, we get that $\rho \equiv \tilde{\pi}$. In particular, $\pi(f) = \int_\Sigma f \, \mathrm{d}E$, for every $f \in C(\Sigma)$. This finishes the proof of the existence assertion.

It remains to argue that E is unique. Assume that E' is a regular spectral measure such that $\pi(f) = \int_\Sigma f \, \mathrm{d}E'$, for every $f \in C(\Sigma)$. Let $f \in B(\Sigma)$. By Lemma 3.4, we can find a net $f_i \in C(\Sigma)$ such that $\|f_i\|_\infty \leqslant \|f\|_\infty$, for all i, and $\int_\Sigma f_i \, \mathrm{d}\mu \to \int_\Sigma f \, \mathrm{d}\mu$, for every regular Borel measure μ. The moreover part of Lemma 6.8 gives that $\pi(f_i) \to \tilde{\pi}(f)$ in the WOT. Since we also have that $\int_\Sigma f_i \, \mathrm{d}E' \to \int_\Sigma f \, \mathrm{d}E'$ in the WOT, we conclude that $\tilde{\pi}(f) = \int_\Sigma f \, \mathrm{d}E'$. Therefore, if $\Delta \in \Omega$, by letting $f = 1_\Delta$ we get that $E(\Delta) = \tilde{\pi}(1_\Delta) = \int_\Sigma 1_\Delta \, \mathrm{d}E' = E'(\Delta)$. ■

7. Borel functional calculus

7.1. Borel functional calculus.

Theorem 7.1 (Borel functional calculus)**.** *Let $a \in \mathbb{B}(H)$ be a normal operator and Ω the σ-algebra of Borel subsets of $\sigma(a)$. Then there exists a regular spectral measure $E : \Omega \to \mathbb{B}(H)$ such that*

$$a = \int_{\sigma(a)} z \; dE$$

For every $f \in B(\sigma(a))$, define $f(a) := \int_{\sigma(a)} f(z) \; dE$. Then the map $B(\sigma(a)) \ni f \to f(a) \in \mathbb{B}(H)$ is a $*$-homomorphism. Moreover, if $f_i \in B(\sigma(a))$ is a net of functions such that $\int_{\sigma(a)} f_i \; d\mu \to 0$, for every Borel regular measure μ on $\sigma(a)$, then $f_i(a) \to 0$ in the WOT.

Proof. By Theorem 5.1, there exists a $*$-homomorphism $\pi : C(\sigma(a)) \to \mathbb{B}(H)$ such that $\pi(z) = a$. The conclusion now follows directly from the spectral theorem 6.2. □

Corollary 7.2. *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra.*

(1) If $a \in M$ is normal, then $f(a) \in M$, for every $f \in B(\sigma(a))$.
(2) M is equal to the norm closure of the linear span of its projections.

Proof. (1) Let $f \in B(\sigma(a))$. Let $f_i \in C(\sigma(a))$ be a net such that $\|f_i\|_\infty \leqslant \|f\|_\infty$, for all i, and $\int_{\sigma(a)} f_i \, \mathrm{d}\mu \to \int_{\sigma(a)} f \, \mathrm{d}\mu$, for every regular Borel measure μ on $\sigma(a)$. By Theorem 7.1, we have that $f_i(a) \to f(a)$ in the WOT. Since $f_i(a) \in C^*(a) \subset M$, we conclude that $f(a) \in M$.

(2) If $a \in M$, then we can write $a = b + ic$,where $b, c \in M$ are self-adjoint. So it suffices to show that any self-adjoint $a \in M$ is belongs to the norm closure of the linear span of projections of M. To this end, let $\varepsilon > 0$ and write $a = \int_{\sigma(a)} z \, \mathrm{d}E$. Then we can find $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and Borel sets $\Delta_1, ..., \Delta_n \subset \sigma(a)$ such that $\|z - \sum_{i=1}^n \alpha_i 1_{\Delta_i}\|_\infty \leqslant \varepsilon$. It follows that $\|a - \sum_{i=1}^n \alpha_i 1_{\Delta_i}(a)\| \leqslant \varepsilon$. Since the projections $1_{\Delta_i}(a)$ belong to M by part (1), we are done. ■

Exercise 7.3. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $a \in M$ with $a \geqslant 0$. Prove that there exist projections $\{p_n\}_{n \geqslant 1}$ such that $a = \|a\| \sum_{n=1}^\infty 2^{-n} p_n$.

Exercise 7.4. Let $a \in \mathbb{B}(H)$ be a positive operator. Prove that a is compact if and only if the projection $1_{[\varepsilon,\infty)}(a)$ is finite dimensional, for every $\varepsilon > 0$.

7.2. **Polar decomposition.**

Definition 7.5. An operator $v \in \mathbb{B}(H)$ is called a **partial isometry** if $\|v(\xi)\| = \|\xi\|$, for all $\xi \in (\ker v)^{\perp}$. In this case, $(\ker v)^{\perp}$ is called the **initial space** of v and the range $\operatorname{ran}(v) = vH$ is called the **final space** of v.

Theorem 7.6 (polar decomposition)**.** *If $a \in \mathbb{B}(H)$, then there exists a unique partial isometry $v \in \mathbb{B}(H)$ with initial space $(\ker a)^{\perp}$ and final space $\overline{ran(a)}$ such that $a = v|a|$, where $|a| = (a^*a)^{\frac{1}{2}}$ is the* **absolute value** *of a.*

Proof. If $\xi \in H$, then $\|a\xi\|^2 = \langle a\xi, a\xi\rangle = \langle a^*a\xi, \xi\rangle = \langle |a|^2\xi, \xi\rangle = \| \, |a|\xi\|^2$. Then the formula $v(|a|\xi) = a\xi$ defines a unitary operator $v : \overline{ran(|a|)} \to \overline{ran(a)}$. We extend v to H by letting $v(\eta) = 0$, for all $\eta \in (\overline{ran(|a|)})^{\perp}$. Then v is a partial isometry such that $v|a| = a$. By definition the final space of v is $\overline{\operatorname{ran}(a)}$, while the initial space of v is $\overline{ran(|a|)} = (\ker |a|)^{\perp} = (\ker a)^{\perp}$ (the second equality follows from the first line of the proof). The uniqueness of v is obvious. ■

Exercise 7.7. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $a \in M$. Let v be the partial isometry provided by Theorem 7.6. Define $l(a)$ to be the projection onto $\overline{ran(a)}$ (the **right support** of a) and $r(a)$ to be the projection onto $(\ker a)^{\perp}$ (the **left support** of a).

(1) Prove that $v \in M$. (**Hint**: prove that v commutes with every unitary element $x \in M'$ and use the bicommutant theorem to deduce that $v \in M$.)
(2) Prove that $l(a) = vv^*$ and $r(a) = v^*v$. Deduce that $l(a), r(a) \in M$.

Exercise 7.8. Let H be a separable Hilbert space. Assume that $a \in \mathbb{B}(H)$ is an operator that is not compact. Prove that there exist $x, y \in \mathbb{B}(H)$ such that $xay = 1$.

Exercise 7.9. Let H be a separable Hilbert space. Prove that any closed two-sided ideal $I \subset \mathbb{B}(H)$ is equal to $\{0\}$, $\mathbb{K}(H)$ or $\mathbb{B}(H)$.

8. Abelian von Neumann algebras

Definition 8.1. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra. A vector $\xi \in H$ is **cyclic** if $\overline{M\xi} = H$.

Remark 8.2. Let (X, μ) be a standard probability space. Then $1 \in L^2(X)$ is a cyclic vector for the abelian von Neumann algebra $L^{\infty}(X) \subset \mathbb{B}(L^2(X))$.

Theorem 8.3. *Let $M \subset \mathbb{B}(H)$ be an abelian von Neumann algebra which admits a cyclic vector $\xi \in H$. Then there exist a compact Hausdorff space X, a regular Borel measure μ on X, and a unitary operator $U : L^2(X) \to H$ such that $M = UL^{\infty}(X)U^*$. Moreover, if H is separable, then X is a compact metrizable space.*

Proof. Let $A \subset M$ be an SOT-dense C*-subalgebra (to prove the moreover assertion, we will make a specific choice for A). Let $X = \Sigma(A)$ be the maximal ideal space of A and $\pi : C(X) \to A \subset \mathbb{B}(H)$ be the inverse of the Gelfand transform (see Theorem 5.1). By the Spectral Theorem 6.2, there exists a regular spectral measure E on X such that $\pi(f) = \int_X f \, \mathrm{d}E$, for every $f \in C(X)$.

Then $\mu(\Delta) = \langle E(\Delta)\xi, \xi\rangle$ defines a Borel regular measure on X such that $\int_X f \, \mathrm{d}\mu = \langle \pi(f)\xi, \xi\rangle$, for every $f \in C(X)$. Thus, for every $f \in C(X)$ we get that

$$\|\pi(f)\xi\|^2 = \langle \pi(f)\xi, \pi(f)\xi\rangle = \langle \pi(f)^*\pi(f)\xi, \xi\rangle = \langle \pi(|f|^2)\xi, \xi\rangle = \int_X |f|^2 \, \mathrm{d}\mu,$$

hence $\|\pi(f)\xi\| = \|f\|_{L^2(X)}$. As $\pi(C(X)) = A$ is SOT-dense in M, we get that $\{\pi(f)\xi | f \in C(X)\}$ is dense in $\overline{M\xi} = H$. Also, Lemma 3.4 implies that $C(X)$ is dense in $L^2(X)$.

The last three facts allow us to define a unitary operator $U : L^2(X) \to H$ by letting

$$U(f) = \pi(f)\xi, \text{ for all } f \in C(X).$$

Now, let $\rho : L^\infty(X) \to \mathbb{B}(L^2(X))$ be the $*$-homomorphism given by $\rho_f(\eta) = f\eta$. Then for every $f, g \in C(X)$ we have that $U\rho_f(g) = U(fg) = \pi(fg)\xi = \pi(f)\pi(g)\xi = \pi(f)U(g)$. Since $C(X)$ is dense in $L^2(X)$ we deduce that $U\rho_f = \pi(f)U$ and therefore $\pi(f) = U\rho_f U^*$, for all $f \in C(X)$.

From this we get that $\pi(C(X)) = U\rho(C(X))U^*$. Since by Lemma 3.4, $\rho(C(X))$ is WOT-dense in $L^\infty(X)$, we conclude that $M = UL^\infty(X)U^*$.

To prove the moreover assertion, assume that H is separable. Then $(\mathbb{B}(H)_1, WOT)$ is a compact metrizable space (see Exercise 1.21) and hence (M_1, WOT) is a compact metrizable space. Let $\{x_n\} \subset M_1$ be a WOT-dense sequence. Define A to be the C*-algebra generated by $\{x_n\}$. Then A is SOT-dense in M. Moreover, $X = \Sigma(A)$ is metrizable. Indeed, one can define a compatible metric by letting $d(\varphi, \varphi') = \sum_{n=1}^\infty \frac{1}{2^n}|\varphi(x_n) - \varphi'(x_n)|$, for every $\varphi, \varphi' \in \sigma(A)$. ■

Theorem 8.4. *Let H be a separable Hilbert space and $M \subset \mathbb{B}(H)$ be an abelian von Neumann algebra. Then M is $*$-isomorphic to $L^\infty(X, \mu)$, where X is a compact metric space and μ is a Borel regular measure on X.*

Proof. By Zorn's lemma, there exists a maximal family $\{\xi_i\}_{i\in I}$ of unit vectors in H such that $\overline{M\xi_i} \perp \overline{M\xi_j}$, for all $i \neq j$. Then $H = \oplus_{i\in I}\overline{M\xi_i}$. To see this, let $\xi \in (\oplus_{i\in I}\overline{M\xi_i})^\perp$. Then for all $x, y \in M, i \in I$ we have $\langle x\xi, y\xi_i\rangle = \langle \xi, x^*y\xi_i\rangle = 0$. Thus $\overline{M\xi} \perp \overline{M\xi_i}$, for all $i \in I$, and the maximality of the family $\{\xi_i\}_{i\in I}$ implies that $\xi = 0$. Also, since H is separable, I must be countable. We reindex the family by $\{\xi_n\}$ and define $\xi = \sum_n 2^{-n}\xi_n$.

We claim that ξ is separating for M. To prove this, let $x \in M$ with $x\xi = 0$. Then $\sum_n 2^{-n}x\xi_n = 0$ and since $x\xi_n \in M\xi_n$, we deduce that $x\xi_n = 0$, for all n. Now, if $y \in M$, then since M is abelian we get that $x(y\xi_n) = y(x\xi_n) = 0$. Thus, $x\eta = 0$, for all $\eta \in \overline{M\xi_n}$ and every n. Therefore, $x\eta = 0$, for all $\eta \in H$, implying that $x = 0$.

Now, denote by p the orthogonal projection from H onto $K := \overline{M\xi}$. Claim 1 from the proof of Theorem 2.4 gives that $p \in M'$. Consequently, for all $x \in M$, we can see xp as an operator on K. We define a $*$-homomorphism $\pi : M \to \mathbb{B}(K)$ by letting $\pi(x) = xp$, for every $x \in M$. Let us show that π is injective. If $xp = 0$, then $x\xi = xp\xi = 0$ and since ξ is separating, it follows that $x = 0$.

Next, we claim that $\pi(M)$ is a von Neumann algebra. Let $\mathcal{M}$ be the closure of $\pi(M)$ in the WOT. Since π is injective, by Lemma 5.14 we get that π is isometric. This implies that $\pi(M)$ is a C*-algebra and $(\pi(M))_1 = \pi(M_1)$. Since π is WOT-continuous and $(M)_1$ is WOT-compact, we derive that $(\pi(M))_1$ is WOT-compact. On the other hand, since $\pi(M)$ is a C*-algebra, Kaplansky's density theorem 5.15 gives that $(\pi(M))_1$ is WOT-dense in $(\mathcal{M})_1$. The last two facts together imply that $(\pi(M))_1 = (\mathcal{M})_1$. Hence, $\pi(M) = \mathcal{M}$ is a von Neumann algebra.

Finally, since $\overline{\pi(M)\xi} = \overline{Mp\xi} = \overline{M\xi} = K$, we get that ξ is a cyclic vector for $\pi(M) \subset \mathbb{B}(K)$. Theorem 8.4 implies that $\pi(M)$ is $*$-isomorphic to $L^\infty(X, \mu)$, for a compact metrizable space X and a regular Borel measure μ on X. Since $\pi(M)$ is $*$-isomorphic to M, we are done. ■

Definition 8.5. A projection p of a von Neumann algebra M is called **minimal** if every projection $q \in M$ such that $0 \leq q \leq p$ satisfies $q \in \{0, p\}$. A von Neumann algebra M is called **diffuse** if any minimal projection is equal to 0.

Corollary 8.6. *Let H be a separable Hilbert space and $M \subset \mathbb{B}(H)$ be a diffuse abelian von Neumann algebra. Then M is $*$-isomorphic to $L^\infty([0, 1], \lambda)$.*

Proof. By Theorem 8.4, M is $*$-isomorphic to $L^\infty(X)$, where (X,μ) is a standard probability space. Since M is diffuse, (X,μ) has no atoms: if $\mu(\{x\})>0$, for some $x\in X$, then $1_{\{x\}}\in L^\infty(X)$ is a non-zero minimal projection. Theorem 3.7 thus implies the conclusion. ∎

Exercise 8.7. Let M be a von Neumann algebra and $p\in M$ be a projection. Prove that p is minimal if and only if $pMp=\mathbb{C}p:=\{\alpha p\mid\alpha\in\mathbb{C}\}$.

Exercise 8.8. Let $M\subset\mathbb{B}(H)$ be a diffuse von Neumann algebra. Let $A\subset M$ be an abelian von Neumann subalgebra which is **maximal abelian**, i.e., satisfies $A'\cap M=A$. Prove that A is diffuse.

9. Decomposition into types for von Neumann algebras

Starting in this section, we explore further the general theory of von Neumann algebras. Recall that any von Neumann algebras is generated by its projections (see Corollary 7.2). It is therefore important to understand better how these projections "interact".

9.1. **Projections.** For a von Neumann algebra M we denote by $\mathcal{P}(M)$ the set of its projections and by $\mathcal{U}(M)$ the group of its unitaries.

Definition 9.1. Let $\{p_i\}_{i\in I}\in\mathbb{B}(H)$ be a family of projections. We denote by

- $\bigvee_{\in I}p_i$ the smallest projection $p\in\mathbb{B}(H)$ such that $p\geqslant p_i$, for all $i\in I$ (equivalently, the orthogonal projection onto the closure of the linear span of $\{p_iH|i\in I\}$).
- $\bigwedge_{i\in I}p_i$ the largest projection $p\in\mathbb{B}(H)$ such that $p\leqslant p_i$, for all $i\in I$ (equivalently, the orthogonal projection onto $\cap_{i\in I}p_iH$.

Proposition 9.2. *If $p_i\in\mathcal{P}(M)$, for all $i\in I$, then $\bigvee_{i\in I}p_i,\bigwedge_{i\in I}p_i\in M$.*

Proof. A projection $p\in\mathbb{B}(H)$ belongs M if and only if p commutes with every $x\in M'$ and if and only if pH is invariant under every $x\in M'$ (see the bicommutant theorem and its proof). ∎

Definition 9.3. Let $M\subset\mathbb{B}(H)$ be a unital von Neumann algebra.

- $\mathcal{Z}(M)=M\cap M'$ is called the **center** of M.
- M is called a **factor** if $\mathcal{Z}(M)=\mathbb{C}\cdot 1$.
- a projection $p\in\mathcal{P}(M)$ is **central** if $p\in\mathcal{Z}(M)$.
- the **central support** of $p\in\mathcal{P}(M)$ is the smallest projection $z(p)\in\mathcal{Z}(M)$ such that $p\leqslant z(p)$.

Lemma 9.4. *$z(p)$ is the orthogonal projection onto $\overline{MpH}$.*

Proof. Let z be the orthogonal projection onto $\overline{MpH}$. Since $pH\subset MpH$, we have that $p\leqslant z$. Since MpH is both M and M' invariant, we get that $p\in M'\cap(M')'=\mathcal{Z}(M)$. Finally, since $p=z(p)p$ we have $MpH=Mz(p)pH=z(p)MpH\subset z(p)H$ and hence $z\leqslant z(p)$. Altogether, $z=z(p)$. ∎

Exercise 9.5. Prove that $z(p)=\bigvee_{u\in\mathcal{U}(M)}upu^*$.

Proposition 9.6. *Let $M\subset\mathbb{B}(H)$ be a von Neumann algebra. Let $p\in\mathcal{P}(M)$ and $p'\in\mathcal{P}(M')$. We denote $pMp=\{pxp|x\in M\}$ and $Mp'=\{xp'|x\in M\}$ and view them as algebras of operators on the Hilbert spaces pH and $p'H$, respectively. Then we have the following:*

1. *$Mp'\subset\mathbb{B}(p'H)$ is a von Neumann algebra and $(Mp')'=p'M'p'$.*
2. *$pMp\subset\mathbb{B}(pH)$ is a von Neumann algebra and $(pMp)'=M'p$.*

Proof. Since (2) clearly implies (1), we only prove (2). To prove (2), note that $M'p \subset (pMp)'$. For the converse inclusion, let $u \in (pMp)'$ be a unitary.

Claim. There exists $\tilde{u} \in M'$ such that $u = \tilde{u}p$ (hence $u \in M'p$).

Let $x_1, ..., x_n \in M$ and $\xi_1, ..., \xi_n \in pH$. Since $u^* x_j^* x_i u = u^* p x_j x_i p u = p x_j^* x_i p$, we have that

$$\|\sum_{i=1}^{n} x_i u \xi_i\|^2 = \sum_{i,j=1}^{n} \langle x_i u \xi_i, x_j u \xi_j \rangle = \sum_{i,j=1}^{n} \langle u^* x_j x_i u \xi_i, \xi_j \rangle = \|\sum_{i=1}^{n} x_i p \xi\|^2 = \|\sum_{i=1}^{n} x_i \xi_i\|^2.$$

Define $\tilde{u} : H \to H$ by $\tilde{u}(\xi) = \sum_{i=1}^{n} x_i u \xi_i$, for $\xi = \sum_{i=1}^{n} x_i \xi_i \in MpH$, and $\tilde{u}(\xi) = 0$, for $\xi \in \overline{MpH}^{\perp}$. Then $\tilde{u}$ is a partial isometry whose left and right supports are equal to $z = z(p)$. Clearly, $\tilde{u}p = u$. To see that $\tilde{u} \in M'$, note that $M = Mz \oplus M(1-z)$. If $y \in M(1-z)$, then $\tilde{u}y = y\tilde{u} = 0$. If $y \in Mz$, then $\tilde{u}y(x\xi) = \tilde{u}(yxp) = yxu\xi = y\tilde{u}(x\xi)$, for all $x \in M$ and $\xi \in pH$, hence $\tilde{y} = y\tilde{u}$. Since every operator in $(pMp)'$ is a linear combination of 4 unitaries, the claim implies that $(pMp)' \subset M'p$. This completes the proof of the equality $(pMp)' = M'p$. To see that pMp is a von Neumann algebra, let $x \in (pMp)''$. If $y \in M'$, then $py = yp \in M'p \subset (pMp)'$ and thus $xy = x(py) = (py)x = y(px) = yx$. Hence $x \in (M')' = M$ and so $x \in pMp$. ■

Corollary 9.7. *$\mathcal{Z}(Mp') = \mathcal{Z}(M)p$ and $\mathcal{Z}(pMp) = \mathcal{Z}(M)p$.*

9.2. **Equivalence of projections.**

Definition 9.8. Two projections $p, q \in M$ are **equivalent** ($p \sim q$) if there exists a partial isometry $v \in M$ such that $p = v^*v$ and $q = vv^*$. We say that p is **dominated by** q (and write $p \prec q$) if $p \sim q'$, for some projection $q' \in M$ with $q' \leqslant q$.

Exercise 9.9. Prove the following:

(1) If $p \sim q$, then $z(p) = z(q)$.
(2) If $p \sim q$ via partial isometry v, then the map $pMp \ni x \to vxv^* \in qMq$ is a $*$-isomorphism.
(3) If $\{p_i\}_{i \in I}, \{q_i\}_{i \in I}$ are families of mutually orthogonal projections and $p_i \sim q_i$, for all $i \in I$, then $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$.
(4) If $p \sim q$ and $z \in M$ is a central projection, then $zp \sim zq$.

Lemma 9.10. *Let M be a von Neumann algebra and $p, q \in \mathcal{P}(M)$. TFAE:*

(1) $pMq \neq \{0\}$.
(2) there exist non-zero projections $p_1, q_1 \in M$ such that $p_1 \leqslant p, q_1 \leqslant q$ and $p_1 \sim q_1$.
(3) $z(p)z(q) \neq 0$.

Proof. (1) $\Rightarrow$ (2) Let $x \in M$ such that $y = pxq \neq 0$. Then $0 \neq p_1 = l(y) \leqslant p, 0 \neq q_1 = r(y) \leqslant q$ and $p_1 \sim q_1$ (see Exercise 7.7).

(2) $\Rightarrow$ (1) If $v \in M$ is such that $p_1 = vv^*$ and $q_1 = v^*v$, then $0 \neq v = pvq \in pMq$.

(1) $\Rightarrow$ (3) If $z(p)z(q) = 0$, then $pxq = pz(p)xz(q)q = pxz(p)z(q)q = 0$, for all $x \in M$.

(3) $\Rightarrow$ (1) If $pMq = \{0\}$, then $p(xq\xi) = 0$, for all $\xi \in H$. Since $z(q)$ is the orthogonal projection onto $\overline{MqH}$, we get that $pz(q) = 0$ and thus $p \leqslant 1 - z(q)$. From this we get that $z(q) \leqslant 1 - z(p)$, hence $z(p)z(q) = 0$. ■

Theorem 9.11 (the comparison theorem)**.** *If $p, q \in \mathcal{P}(M)$, then there exists a projection $z \in \mathcal{Z}(M)$ such that $pz \prec qz$ and $q(1-z) \prec p(1-z)$.*

Proof. By Zorn's lemma, there exist maximal families of mutually orthogonal projections $\{p_i\}_{i\in I}, \{q_i\}_{i\in I}$ such that $p_i \leqslant p, q_i \leqslant q$ and $p_i \sim q_i$, for all $i \in I$. Put $p_1 = \sum_{i\in I} p_i$, and $q_1 = \sum_{i\in I} q_i$. Then $p_1 \sim q_1$. Also, let $p_2 = p - p_1$ and $q_2 = q - q_1$.

Since p_2, q_2 do not have equivalent non-zero subprojections, Lemma 9.10 implies that $z(p_2)z(q_2) = 0$. Thus, if we let $z = z(q_2)$, then $p_2 z = 0$ and $q_2(1-z) = 0$. The conclusion now follows since

$$pz = \sum_{i\in I} p_i z + p_2 z = \sum_{i\in I} p_i z \sim \sum_{i\in I} q_i z \prec \sum_{i\in I} q_i z + q_2 z = qz$$

and similarly $q(1-z) \prec p(1-z)$. ■

Corollary 9.12. *If M is a factor and $p, q \in \mathcal{P}(M)$, then $p \prec q$ or $q \prec p$.*

9.3. Classification into types.

Definition 9.13. A projection $p \in M$ is called:

(1) **abelian** if pMp is abelian.
(2) **finite** if whenever $q \in M$ is a projection such that $q \leqslant p$ and $q \sim p$, then $q = p$.

Remark 9.14. Every abelian projection is finite. Also, a subprojection of an abelian (resp. finite) projection is abelian (resp. finite).

Definition 9.15. A unital von Neumann algebra $M \subset \mathbb{B}(H)$ is called

- **finite** if $1 \in M$ is finite.
- of **type I** if any non-zero central projection contains a non-zero abelian subprojection.
- of **type II** if it has no abelian projections and any non-zero central projection contains a non-zero finite subprojection.
- of **type III** if it contains no non-zero finite projection.
- of **type $\mathbf{I_{fin}}$** if it is of type I and finite.
- of **type $\mathbf{I_\infty}$** if it is of type I and not finite.
- of **type $\mathbf{II_1}$** if it is of type II and finite.
- of **type $\mathbf{II_\infty}$** if it is of type II and not finite.

Remark 9.16. M is finite iff any isometry is a unitary, i.e., $v^*v = 1 \Rightarrow vv^* = 1$.

Exercise 9.17. Prove that $\mathbb{B}(H)$ is finite if and only if H is finite dimensional.

Exercise 9.18. Let M be a von Neumann algebra and $p \in \mathcal{P}(M)$.

(1) Prove that if $p \in M$ is abelian, then $Mz(p)$ is of type I.
(2) Prove that if $p \in M$ is finite, then $Mz(p)$ is semifinite: every non-zero central projection q of $Mz(p)$ contains a non-zero finite subprojection.

Theorem 9.19. *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra. Then there exist unique central projections $z_1, ..., z_5 \in \mathcal{Z}(M)$ with $\sum_{i=1}^5 z_i = 1$ and $Mz_1, Mz_2, Mz_3, Mz_4, Mz_5$ are von Neumann algebras of the type $I_{fin}, I_\infty, II_1, II_\infty, III$, respectively.*

For a proof see [Co99, Theorem 48.16] or the next exercise.

Exercise 9.20. Let $p, q, r \in \mathcal{Z}(M)$ be the maximal projections such that Mp is of type I, Mq is of type II, and r is finite projection. Define $z_1 = pr, z_2 = p(1-r), z_3 = qr, z_4 = q(1-r)$ and $z_5 = 1 - (p+q)$. Use Exercise 9.18 to prove that $z_1, .., z_5$ satisfy the conclusion of Theorem 9.19.

9.4. **von Neumann algebras of type I.**

Definition 9.21. Let $M \subset \mathbb{B}(H)$ and $N \subset \mathbb{B}(K)$ be von Neumann algebras. For $x \in M, y \in N$ we define $x \otimes y \in \mathbb{B}(H \otimes K)$ by letting $(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$, for all $\xi \in H, \eta \in K$.

The **tensor product von Neumann algebra** $M\overline{\otimes}N \subset \mathbb{B}(H \otimes K)$ is defined as the closure of the linear span of $\{x \otimes y | x \in M, y \in N\}$ in the SOT.

Exercise 9.22. Let (X, μ) be a standard probability space. Show that $\mathbb{M}_n(L^\infty(X, \mu)) = \mathbb{M}_n(\mathbb{C})\overline{\otimes}L^\infty(X)$ is a type I_{fin} von Neumann algebra. Show that if K is an infinite dimensional Hilbert space, then $\mathbb{B}(K)\overline{\otimes}L^\infty(X)$ is a type I_∞ von Neumann algebra.

Remark 9.23. As it turns out, any type I von Neumann algebra M is isomorphic to a direct sum of algebras of type I_{fin} and I_∞ as in the previous exercise (see [Co99, Section 50] for a proof). Here, limit ourselves to giving a proof of this fact in the case M is a factor.

Theorem 9.24. *Let M be a von Neumann algebra. Assume that $\{p_i\}_{i \in I}$ is a family of mutually orthogonal projections such that $\sum_{i \in I} p_i = 1$. Let $i_0 \in I$ and put $p = p_{i_0}$. Then M is $*$-isomorphic to $\mathbb{B}(\ell^2(I))\overline{\otimes}pMp$.*

Proof. Denote by $\{\delta_i\}_{i \in I}$ the canonical orthonormal basis of $\ell^2 I$. For $i, j \in I$, we let $e_{i,j} \in \mathbb{B}(\ell^2 I)$ be the "elementary" operator given by $e_{i,j}\delta_k = \delta_{j,k}\delta_i$, for all $k \in I$.

For $i \in I$, let $v_i \in M$ be a partial isometry such that $v_i^* v_i = p$ and $v_i v_i^* = p_i$. We take $v_{i_0} = p$. We define $U : H \to \ell^2 I \otimes pH$ by letting $U(\xi) = \sum_{i \in I} \delta_i \otimes v_i^* \xi$. Since $\sum_{i \in I} \|v_i^* \xi\|^2 = \sum_{i \in I} \|p_i \xi\|^2 = \|\xi\|$, for all $\xi \in H$, it follows that U is a unitary.

We claim $UMU^* = \mathbb{B}(\ell^2(I))\overline{\otimes}pMp$. To this end, note that $U^*(\sum_{i \in I} \delta_i \otimes \xi_i) = \sum_{i \in I} v_i \xi_i$, for all $\xi_i \in pH$. Using this fact, one checks the following:

- $UxU^* = e_{i_0,i_0} \otimes x$, for all $x \in pMp$.
- $Uv_iU^* = e_{i,i_0} \otimes p$, for all $i \in I$.

Let $\mathcal{A} \subset M$ be the $*$-algebra generated by $\{v_i\}_{i \in I} \cup pMp$. Let $\mathcal{B} \subset \mathbb{B}(\ell^2(I))\overline{\otimes}pMp$ be the $*$-algebra generated by $\{e_{i,i_0}\}_{i \in I} \cup \{e_{i_0,i_0} \otimes x | x \in pMp\}$. The last formulae imply that $U\mathcal{A}U^* = \mathcal{B}$.

Let $x \in M$ and for $F \subset I$ finite, denote $p_F = \sum_{i \in F} p_i$. Since $p_F \to 1$, we get that $p_F x p_F \to x$, in the SOT. On the other hand, since $p_F x p_F = \sum_{i,j \in F} v_i v_i^* x v_j v_j^* = \sum_{i,j \in F} v_i (v_i^* x v_j) v_j^*$ and $v_i^* x v_j \in pMp$, we get that $p_F x p_F \in \mathcal{A}$. This shows that $\mathcal{A}$ is SOT-dense in M. Similarly, we get that $\mathcal{B}$ is SOT-dense in $\mathbb{B}(\ell^2 I)\overline{\otimes}pMp$. The claim and the theorem are now proven. ■

Exercise 9.25. Let K be a Hilbert space. Prove that $\mathbb{B}(K)$ is a factor of type I.

Corollary 9.26. *Any factor M of type I is $*$-isomorphic to $\mathbb{B}(K)$, for some Hilbert space K.*

Proof. Let $p \in M$ be a non-zero abelian projection. Then pMp is both abelian and a factor. Therefore, $pMp = \mathbb{C} \cdot p$. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal projections in M that are equivalent to p. Put $q = 1 - \sum_{i \in I} p_i$. We claim that $q = 0$. Indeed, if $q \neq 0$, then by Corollary 9.12 we have that either (1) $p \prec q$ or (2) $q \prec p$. Now, (1) contradicts the maximality of $\{p_i\}_{i \in I}$, while (2) implies that there exists a projection $q' \leqslant p$ such that $q' \sim q$. Since $pMp = \mathbb{C}p$, it follows that $q' = p$, contradicting again the maximality of $\{p_i\}_{i \in I}$. Since $q = 0$, the conclusion is a consequence of Theorem 9.24. ■

10. Tracial von Neumann algebras

10.1. Tracial von Neumann algebras.

Definition 10.1. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra. A state $\varphi : M \to \mathbb{C}$ is called

- [2]**normal** if $\varphi(\bigvee_{i\in I} p_i) = \sup_{i\in I} \varphi(p_i)$, for any increasing net of projections $\{p_i\}_{i\in I}$.
- **tracial** if $\varphi(xy) = \varphi(yx)$, for all $x, y \in M$.

Definition 10.2. A von Neumann algebra M is called **tracial** if it admits a normal, faithful, tracial state $\tau : M \to \mathbb{C}$. (In short, we will say that the pair (M, τ) is a tracial von Neumann algebra.)

Examples 10.3. (of tracial von Neumann algebras)

(1) $L^\infty(X)$ is a tracial von Neumann algebra with the trace given by $\tau(f) = \int_X f \, \mathrm{d}\mu$.
(2) $\mathbb{M}_n(\mathbb{C})$ is a tracial von Neumann algebra with the normalized trace $\tau([a_{i,j}]) = \frac{1}{n}\sum_{i=1}^n a_{i,i}$.
(3) More generally, $\mathbb{M}_n(L^\infty(X))$ is a tracial von Neumann algebra, where $\tau([f_{i,j}]) = \frac{1}{n}\sum_{i=1}^n \int_X f_{i,i} \, \mathrm{d}\mu$.

Remark 10.4. Any tracial von Neumann algebra M is finite. To see this, let $v \in M$ such that $v^*v = 1$. Then vv^* is a projection, hence $1 - vv^*$ is a projection. Since $\tau(1 - vv^*) = \tau(v^*v - vv^*) = 0$ and τ is faithful, we get that $vv^* = 1$.

Theorem 10.5. *Any finite von Neumann algebra M on a separable Hilbert space H is tracial. Any II_1 factor is a tracial von Neumann algebra.*

Remark 10.6. If $M \subset \mathbb{B}(H)$ is a finite von Neumann algebra on an arbitrary Hilbert space H, then there exists a **normal center-valued trace** $\Psi : M \to \mathcal{Z}(M)$ (see [KR97, Chapter 8], for a constructive proof, and [Co99, Section 55], for a proof based on the Ryll-Nardzewski fixed point theorem). In particular, any II_1 factor M is tracial. Now, if H is separable, then $\mathcal{Z}(M)$ is isomorphic to $L^\infty(X)$, for a standard probability space (X, μ) by Theorem 8.4. Then $\tau(T) = \int_X \Psi(T) \, \mathrm{d}\mu$ defines a normal, faithful, tracial state on M.

Exercise 10.7. Let M be a II_1 factor with a faithful normal tracial state τ. Prove that two projections $p.q \in M$ is equivalent if and only if $\tau(p) = \tau(q)$. (**Hint for ($\Leftarrow$):** use Corollary 9.12).

10.2. The standard representation.

A von Neumann algebra can sit in many ways inside $\mathbb{B}(H)$. In this section, we show that any tracial von Neumann algebra (M, τ) has a canonical representation on a Hilbert space. This representation is a particular case of the GNS construction.

Endow M with the scalar product $\langle x, y\rangle = \tau(y^*x)$. Define $L^2(M)$ to be the closure of M with respect to the 2-norm $\|x\|_2 = \sqrt{\tau(x^*x)}$. Let $M \ni x \to \hat{x} \in L^2(M)$ be the canonical embedding. Since $\|xy\|_2^2 = \tau(y^*x^*xy) \leqslant \|x^*x\|\tau(y^*y) = \|x\|^2\|y\|_2^2$, letting $\pi(x)(\hat{y}) = \widehat{xy}$, for all $x, y \in M$, defines a $*$-homomorphism $\pi : M \to \mathbb{B}(L^2(M))$. π is called the **standard representation** of M.

Remark 10.8. (1) $\hat{1} \in L^2(M)$ is a cyclic and separating vector for $\pi(M)$.
(2) τ can be recovered as a vector state: $\tau(x) = \langle \pi(x)\hat{1}, \hat{1}\rangle$.
(3) π is injective, hence isometric. Therefore, $\pi(M)$ is a C*-algebra.

Theorem 10.9. *$\pi(M)$ is a von Neumann algebra.*

Lemma 10.10. *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra and $\varphi : M \to \mathbb{C}$ be a normal state. Then $\varphi_{|(M)_1}$ is WOT-continuous.*

[2]A positive linear map $\varphi : M \to N$ between von Neumann algebras is **normal** if $\varphi(A_i) \to \varphi(A)$ (SOT) for any increasing net (A_i) in M such that $A_i \to A$ (SOT), see [Co99, Definition 46.1]. By [Co99, Theorem 46.4], for states, this notion is equivalent to the definition given here.

Remark 10.11. Although we will not need this fact, note that a state $\varphi : M \to \mathbb{C}$ is normal if and only if it is weak*-continuous (see [Co99, Theorem 46.4]). (The weak*-topology on M arises from identifying $\mathbb{B}(H)$ with the dual of $\mathbb{B}_1(H)$ = the ideal of trace class operators).

Proof. We begin with a claim:

Claim 1. If $p \in M$ is a non-zero projection, then there exists a non-zero projection $r \leqslant p$ and $\xi \in H$ such that $\varphi(x) \leqslant \langle x\xi, \xi\rangle$, for all $x \in rMr$ with $x \geqslant 0$.

(Any normal functional is "locally" (i.e. on a corner of M) dominated by a vector functional).

Proof of Claim 1. To prove the claim, choose $\xi \in H$ such that $\varphi(p) < \langle p\xi, \xi\rangle$. Define $\psi : M \to \mathbb{C}$ by letting $\psi(x) = \langle x\xi, \xi\rangle$. Then $\varphi(p) < \psi(p)$. Let $\{q_i\}_{i\in I}$ be a maximal family of mutually orthogonal projections in pMp such that $\psi(q_i) < \varphi(q_i)$, for all $i \in I$. Let $q = \sum_{i\in I} q_i$. Since ψ is normal, it is completely additive: $\psi(q) = \psi(\bigvee_{F\subset I \text{ finite}} \sum_{i\in F} q_i) = \sup_{F\subset I \text{ finite}} \psi(\sum_{i\in F} q_i) = \sum_{i\in I} \psi(q_i)$. We deduce that $\psi(q) < \varphi(q)$. As a consequence, $r = p - q \neq 0$.

Moreover, the maximality of $\{q_i\}_{i\in I}$ implies that $\varphi(s) \leqslant \psi(s)$, for any projection $s \in M$ with $s \leqslant r$. Now, let $x \in rMr$ with $x \geqslant 0$. By Exercise 7.3, for every $\varepsilon > 0$, there exist $\alpha_1, ..., \alpha_n \in [0, \infty)$ and projections $p_1, ..., p_n \in rMr$ such that $\|x - \sum_{i=1}^n \alpha_i p_i\| < \varepsilon$. Since both φ and ψ are norm continuous (see Exercise 5.10), we deduce that $\varphi(x) \leqslant \psi(x)$. This proves Claim 1. $\square$

By Claim 1, we can find a family $\{p_i\}_{i\in I}$ of mutually orthogonal projections in M and vectors $\{\xi_i\}_{i\in I}$ such that $\sum_{i\in I} p_i = 1$ and $\varphi(x) \leqslant \langle x\xi_i, \xi_i\rangle$, for all $x \in p_iMp_i$ with $x \geqslant 0$ and all $i \in I$.

To finish the proof we also need a "Hilbert space trick":

Claim 2. For every $i \in I$ we can find $\eta_i \in H$ such that $\varphi(xp_i) = \langle xp_i\xi_i, \eta_i\rangle$, for all $x \in M$.

Proof of Claim 2. If $x \in M$, then Cauchy-Schwarz (see Exercise 5.9) and Claim 1 give that:

$$|\varphi(xp_i)|^2 \leqslant \varphi(p_ix^*xp_i) \leqslant \langle p_ix^*xp_i\xi_i, \xi_i\rangle.$$

Denote by $K \subset H$ the closure of $\{xp_i\xi_i | x \in M\}$. Then $K \ni xp_i\xi_i \to \varphi(xp_i) \in \mathbb{C}$ is a well-defined bounded linear functional. Applying Riesz's representation theorem now implies Claim 2. $\square$

Finally, let $x_k \in (M)_1$ be a net such that $x_k \to 0$ in the WOT. We want to show that $\varphi(x_k) \to 0$. Towards this, fix $\varepsilon > 0$. Since $\sum_{i\in I} \varphi(p_i) = \varphi(\sum_{i\in I} p_i) = \varphi(1) = 1$, we can find a finite set $F \subset I$ such that $\sum_{i\in F} \varphi(p_i) > 1 - \varepsilon$. Thus, letting $p_0 = \sum_{i\in I\setminus F} p_i$, then $\varphi(p_0) < \varepsilon$.

Let $x \in (M)_1$. Then Cauchy-Schwarz gives that $|\varphi(xp_0)| \leqslant \varphi(x^*x)\varphi(p_0) \leqslant \|x\|^2\varphi(p_0) < \varepsilon$. Since $\sum_{i\in F} p_i + p_0 = 1$, we get that

$$|\varphi(x)| \leqslant \sum_{i\in F} |\varphi(xp_i)| + |\varphi(xp_0)| \leqslant \sum_{i\in F} |\langle xp_i\xi_i, \eta_i\rangle| + \varepsilon.$$

Thus, $\limsup_k |\varphi(x_k)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we are done. ■

Proof of Theorem 10.9. Since $\pi(M)$ is a C*-algebra, in order to show that it is a von Neumann algebra, by Kaplansky's density theorem it suffices to prove that $(\pi(M))_1$ is SOT-closed. Let $\{\pi(x_i)\}_i \in \pi(M)_1$ be a net such that $\pi(x_i) \to T \in \mathbb{B}(L^2(M))$ in the SOT.

Since π is an isometry, $x_i \in (M)_1$. Since $(M)_1$ is WOT-compact, after passing to a subnet we may assume that $\{x_i\}_i$ converges to some $x \in (M)_1$ in the WOT. Let $y_1, y_2 \in M$. Then $y_2^*x_iy_1 \to y_2^*xy_1$ in the WOT. Since τ is normal, Lemma 10.10 implies that its restriction to $(M)_1$ is WOT-continuous. In particular, we get that $\lim_i \tau(y_2^*x_iy_1) = \tau(y_2^*xy_1)$ and therefore

$$\lim_i \langle \pi(x_i)(\hat{y_1}), \hat{y_2}\rangle = \lim_i \tau(y_2^*x_iy_1) = \tau(y_2^*xy_1) = \langle \pi(x)(\hat{y_1}), \hat{y_2}\rangle.$$

Since $M \subset L^2(M)$ is dense, we conclude that $\pi(x_i) \to \pi(x)$ in the WOT, hence $T = \pi(x) \in \pi(M)_1$. This shows that $\pi(M)_1$ is SOT-closed and finishes the proof. ■

Exercise 10.12. Let (M, τ) be a tracial von Neumann algebra. Prove that a state $\varphi : M \to \mathbb{C}$ is normal if and only if for every $\varepsilon > 0$, we can find $y \in M$ such that $\sup_{x \in (M)_1} |\varphi(x) - \tau(xy)| < \varepsilon$.

10.3. **The commutant of the standard representation.** We next identify the commutant of M in the standard representation and show that it is anti-isomorphic to M. We start by defining $J : L^2(M) \to L^2(M)$ as follows: $J(\hat{x}) = \hat{x^*}$. Then J is a conjugate linear unitary involution: $J(\alpha\hat{x} + \beta\hat{y}) = \overline{\alpha}J(\hat{x}) + \overline{\beta}J(\hat{y})$, $\langle J(\hat{x}), J(\hat{y})\rangle = \langle \hat{y}, \hat{x}\rangle$, if $\alpha, \beta \in \mathbb{C}$, $x, y \in M$, and $J^2 = I$.

Theorem 10.13. *$M' = JMJ$.*

Proof. Denote $H = L^2(M)$. Notice that $\{x\hat{1} | x \in M\}$ is dense in H and $J(x\hat{1}) = x^*\hat{1}$, for all $x \in M$. Using these properties for every $x, y, z \in M$ we get that

$$JxJy(z\hat{1}) = JxJ(yz\hat{1}) = Jx(z^*y^*\hat{1}) = J(xz^*y^*\hat{1}) = yzx^*\hat{1} = yJ(xz^*\hat{1}) = yJxJ(z\hat{1}).$$

This shows that $JMJ \subset M'$.

In particular, we get that $\{x'\hat{1} | x' \in M'\} \supset \{JxJ\hat{1} | x \in M\} = \{\hat{x^*} | x \in M\} = \{\hat{x} | x \in M\}$, which implies that $\{x'\hat{1} | x' \in M'\}$ is dense in H. Further, if $x' \in M'$, then for every $y \in M$ we have that

$$\langle Jx\hat{1}, y\hat{1}\rangle = \langle Jy\hat{1}, x\hat{1}\rangle = \langle x^*y^*\hat{1}, \hat{1}\rangle = \langle y^*x^*\hat{1}, \hat{1}\rangle = \langle x^*\hat{1}, y\hat{1}\rangle.$$

This shows that $Jx\hat{1} = x^*\hat{1}$, for all $x \in M'$. Altogether, have shown that the two properties satisfied by M are also verified by M'. Thus, we deduce that $JM'J \subset M'' = M$ and hence $JMJ = M'$. ■

10.4. **Hilbert modules.** Next, we address the following question: on what Hilbert spaces other than $L^2(M)$ can a tracial von Neumann algebra M be represented? If $\pi : M \to \mathbb{B}(H)$ is an isometric $*$-homomorphism, then $\pi(M)$ is a von Neumann algebra iff π is normal. This fact motivates the following definition:

Definition 10.14. Let M be a von Neumann algebra. A $*$-homomorphism $\pi : M \to \mathbb{B}(H)$ is [3]**normal** if the linear functional $M \ni x \to \langle \pi(x)\xi, \xi\rangle \in \mathbb{C}$ is normal, for every $\xi \in H$. A **left Hilbert M-module** is a Hilbert space M together with a unital normal $*$-homomorphism $\pi : M \to \mathbb{B}(H)$. (Note that defining $x \cdot \xi = \pi(x)\xi$ makes H a left M-module.)

Exercise 10.15. Let $\varphi : M \to \mathbb{C}$ be a state. Let $\pi_\varphi : M \to \mathbb{B}(H_\varphi)$ the GNS $*$-homomorphism. Prove that φ is normal if and only if π_φ is normal.

Exercise 10.16. Prove that there exists a non-normal state $\varphi : L^\infty([0,1], \lambda) \to \mathbb{C}$. Deduce the existence of $*$-homomorphisms $\pi : L^\infty([0,1], \lambda) \to \mathbb{B}(H)$ which are not normal.

Theorem 10.17. *If H is a left Hilbert M-module, there exists a family of projections $\{p_i\}_{i \in I}$ in M such that $H \cong \oplus_{i \in I} L^2(M)p_i$. More precisely, there exists a unitary operator $U : H \to \oplus_{i \in I} L^2(M)p_i$ such that $U(x \cdot \xi) = x \cdot U(\xi)$, for all $x \in M$ and $\xi \in H$.*

Lemma 10.18 (Radon-Nikodym). *Let $\varphi : M \to \mathbb{C}$ be a linear functional such that $0 \leqslant \varphi(x) \leqslant \tau(x)$, for all $x \in M$ with $x \geqslant 0$. Then there exists $y \in M$ such that $0 \leqslant y \leqslant 1$ and $\varphi(x) = \tau(xy)$, for all $x \in M$.*

[3]Let (A_i) be an increasing net in M such that $A_i \to A$ (SOT). Then for $\xi \in H$, since $x \mapsto \langle \pi(x)\xi, \xi\rangle$ is normal, we have that $\langle \pi(A_i)\xi, \xi\rangle \to \langle \pi(A)\xi, \xi\rangle$ and hence $\pi(A_i) \to \pi(A)$ (SOT) by [Co99, Proposition 43.1]. Thus, the notion considered here is the same as the usual notion of normality from [Co99, Definition 46.1].

Proof, Cauchy-Schwarz gives that $|\varphi(y^*x)|^2 \leqslant \varphi(x^*x)\varphi(y^*y) \leqslant \tau(x^*x)\tau(y^*y) = \|x\|_2^2\|y\|_2^2$, for all $x, y \in M$. In particular, $|\varphi(x)| \leqslant \|x\|_2 = \|\hat{x}\|_2$, for all $x \in M$. By Riesz's representation theorem we find $\xi \in L^2(M)$ such that $\varphi(x) = \langle \hat{x}, \xi\rangle$, for all $x \in M$. Next, for $y \in M$, we get that

$$\|y\xi\|_2 = \sup_{x\in M, \|x\|_2\leqslant 1} |\langle \hat{x}, y\xi\rangle| = \sup_{x\in M, \|x\|_2\leqslant 1} |\langle \widehat{y^*x}, \xi\rangle| = \sup_{x\in M, \|x\|_2\leqslant 1} |\varphi(y^*x)| \leqslant \|y\|_2 = \|\hat{y}\|_2.$$

This inequality implies that there exists $T \in \mathbb{B}(L^2(M))$ such that $T(\hat{y}) = y\xi$, for all $y \in M$. Then $xT(\hat{y}) = xy\xi = Tx(\hat{y})$, for all $x \in M$. We deduce that $T \in M'$, which by Theorem 10.13 implies that $T \in JMJ$. Hence there exists $y \in M$ such that $\xi = T(\hat{1}) = JyJ(\hat{1}) = \hat{y^*}$. Thus, we have that $\varphi(x) = \tau(xy)$, for all $x \in M$. It is left as an exercise to show that $0 \leqslant y \leqslant 1$. ■

Exercise 10.19. Let (M, τ) be a tracial von Neumann algebra. Assume that $y \in M$ satisfies $0 \leqslant \tau(xy) \leqslant \tau(x)$, for all $x \in M$ with $x \geqslant 0$. Prove that $0 \leqslant y \leqslant 1$.

Proof of Theorem 10.17. The proof relies on the following claim:

Claim. Let $\xi \in H \setminus \{0\}$. Then we can find non-zero projections $q, p \in M$ such that $\overline{Mq\xi} \cong L^2(M)p$, as left Hilbert M-modules.

Proof of the claim. To prove the claim, define $\varphi : M \to \mathbb{C}$ by letting $\varphi(x) = \langle x\xi, \xi\rangle$, for every $x \in M$. Since φ is normal, we can find a non-zero projection $r \in M$ such that $\varphi(q) > 0$ for every non-zero projection $q \in rMr$. Indeed, if no such r exists, then any maximal familiy $\{p_i\}_{i\in I}$ of mutually orthogonal projections such that $\varphi(p_i) = 0$, for all $i \in I$, would necessarily satisfy that $\sum_{i\in I} p_i = 1$. Since φ is normal we would get that $\varphi(1) = 0$ which contradicts $\xi \neq 0$.

Let $c > 0$ such that $\varphi(r) < c\tau(r)$. Note that φ and $c\tau$ are both normal positive linear functionals on rMr. Then the proof of Claim 1 in the proof of Lemma 10.10 implies that we can find a projection $q \in rMr$ such that $\varphi(x) \leqslant c\tau(x)$, for all $x \in qMq$ with $x \geqslant 0$.

By applying Lemma 10.18, we can find $y \in qMq$ such that $0 \leqslant y \leqslant c$ and $\varphi(x) = \tau(xy)$, for all $x \in qMq$. Let $z \in qMq$ such that $z \geqslant 0$ and $z^2 = y$. If $x \in M$, then since $qxq \in qMq$, we get that

$$\langle x(q\xi), q\xi\rangle = \langle qxq\xi, \xi\rangle = \varphi(qxq) = \tau(qxqy) = \tau(xy) = \tau(xz^2) = \langle x\hat{z}, \hat{z}\rangle_{L^2(M)}.$$

From this, we get that $\|x(q\xi)\| = \|x\hat{z}\|_2$, for all $x \in M$. We conclude that $\theta : \overline{Mq\xi} \to \overline{M\hat{z}} \subset L^2(M)$ given by $\theta(x(q\xi)) = x\hat{z}$ extends to a unitary operator. It follows that $\overline{Mq\xi} \cong \overline{M\hat{z}}$, as left Hilbert M-modules. The proof of the claim is done modulo the following exercise:

Exercise 10.20. Let $z \in M$ with $z \geqslant 0$ and denote by p the support projection of z. Then we have that $\overline{M\hat{z}} \cong \overline{M\hat{p}} = L^2(M)p$, as left Hilbert M-modules. □

Finally, let $\{H_i\}_{i\in I}$ be a maximal family of mutually orthogonal left Hilbert M-sub-modules of H such that for every $i \in I$, there exists $p_i \in \mathcal{P}(M)$ with $H_i \cong L^2(M)p_i$. In order to finish the proof, it is enough to show that $H = \oplus_{i\in I} H_i$. Assume by contradiction that there is $\xi \in H \setminus \{0\}$ such that $\xi \in (\oplus_{i\in I} H_i)^\perp$. Then the claim provides non-zero projections $p, q \in M$ such that $K := \overline{Mq\xi} \cong L^2(M)p$. Since $K \subset \overline{M\xi} \subset (\oplus_{i\in I} H_i)^\perp$, this contradicts the maximality of $\{H_i\}_{i\in I}$. ■

Theorem 10.21. *Let (M, τ) be a tracial von Neumann and $\{p_i\}_{i\in I}$, $\{q_j\}_{j\in J}$ be projections in M. Assume that either*

(1) there exists a linear injective map $L : \oplus_{i\in I} Mp_i \to \oplus_{j\in J} Mq_j$ such that $L(x \cdot \xi) = x \cdot L(\xi)$, for all $x \in M$ and $\xi \in \oplus_{i\in I} Mp_i$, or

(2) there exists an injective bounded operator (e.g., an isometry) $T : \oplus_{i\in I} L^2(M)p_i \to \oplus_{j\in J} L^2(M)q_j$ such that $T(x \cdot \xi) = x \cdot T(\xi)$, for all $x \in M$ and $\xi \in \oplus_{i\in I} L^2(M)p_i$.

Then $\sum_{i\in I}\tau(p_i)\leq\sum_{j\in J}\tau(q_j)$.

Proof. Since $\sum_{i\in I}\tau(p_i)=\sup\{\sup_{i\in F}\tau(p_i)\mid F\subset I\text{ finite}\}$, we may assume that I is finite. We may also assume that J is countable. Suppose therefore that $J=\{1,2,...,n\}$, where $n\in\mathbb{N}\cup\{\infty\}$.

Let us first show that (1) implies (2). If L is as in (1), then we can find $\alpha_{i,j}\in p_iMq_j$ such that $L(\oplus_{i\in I}x_ip_i)=\oplus_{j\in J}(\sum_{i\in I}x_i\alpha_{i,j})$, for all $(x_i)_{i\in I}\subset M$. Since $\|\sum_{i\in I}x_i\alpha_{i,j}\|_2\leq\max_{i\in I}\|\alpha_{i,j}\|\sum_{i\in I}\|x_ip_i\|_2$, if we let $c_{i,j}:=2^j(\max_{i\in I}\|\alpha_{i,j}\|+1)$, then $T:\oplus_{i\in I}L^2(M)p_i\to\oplus_{j\in J}L^2(M)q_j$ given by

$$T(\oplus_{i\in I}x_ip_i)=\oplus_{j\in J}(c_{i,j}^{-1}\sum_{i\in I}x_i\alpha_{i,j})$$

is an injective bounded operator satisfying (2).

Now, assuming that (2) holds, let $K=I\sqcup J$ and put $p_i=q_j=0$, for all $i\in K\setminus I$ and $j\in K\setminus J$. Put $H=L^2(M)\bar{\otimes}\ell^2(K)$ and view $M\subset\mathbb{B}(H)$. Embed $\oplus_{k\in K}L^2(M)p_k,\oplus_{k\in K}L^2(M)q_k$ into H, in the natural way. Then T extends to a bounded injective operator on H which commutes with M. In other words, $T\in M'\cap\mathbb{B}(H)=JMJ\bar{\otimes}\mathbb{B}(\ell^2(K))$. Let V be the partial isometry in the polar decomposition of T. By Exercise 7.7, $V\in JMJ\bar{\otimes}\mathbb{B}(\ell^2(K))$ and the left and right support projections of T satisfy $l(T)=VV^*$ and $r(T)=V^*V$. Since $T_{|\oplus_{k\in K}L^2(M)p_k}$ is injective we have that $r(T)=\oplus_{k\in K}Jp_kJ$. Also, by definition we have that $l(T)\leq\oplus_{k\in K}Jq_kJ$. Write V in matrix form as $V=(JV_{k,l}J)_{k,l\in K}$, where $V_{k,l}\in M$ for all $k,l\in K$. Then, for every $k\in K$, we have that $p_k=\sum_{l\in K}V_{l,k}^*V_{l,k}$ and $q_k\geq\sum_{l\in K}V_{k,l}V_{k,l}^*$. Thus, we conclude that

$$\sum_{k\in K}\tau(p_k)=\sum_{k,l\in K}\tau(V_{l,k}^*V_{l,k})=\sum_{k,l\in K}\tau(V_{k,l}V_{k,l}^*)\leq\sum_{k\in K}\tau(q_k).$$

■

Definition 10.22. Let (M,τ) be a tracial von Neumann algebra. We define the **dimension** of a left Hilbert M-module H as $\dim_M H:=\sum_{i\in I}\tau(p_i)$, where $\{p_i\}_{i\in I}\subset M$ is any family of projections such that $H\cong\oplus_{i\in I}L^2(M)p_i$. (By Theorem 10.21, $\dim_M H$ is independent of the choices made.)

10.5. **Conditional expectation.**

Definition 10.23. Let M be a von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. A linear map $E:M\to B$ is called a **conditional expectation** if it satisfies the following:

(1) $E(b)=b$, for every $b\in B$.
(2) $E(x)\geqslant 0$, for every $x\in M$ with $x\geqslant 0$.
(3) $E(b_1xb_2)=b_1E(x)b_2$, for every $b_1,b_2\in B$ and $x\in M$.

Proposition 10.24. *Let (M,τ) be a tracial von Neumann algebra and $B\subset M$ be a von Neumann subalgebra. Then there exists a unique trace preserving conditional expectation $E:M\to B$.*

Proof. Let $e_B:L^2(M)\to L^2(B)$ be the orthogonal projection, where $L^2(B)$ denotes the $\|.\|_2$-closure of $\{\hat{b}|b\in B\}$. If $x\in M$ and $b\in B$, then $be_B(\hat{x})=e_B(\hat{bx})$ and hence

$$\|be_B(\hat{x})\|_2=\|e_B(\widehat{bx})\|_2\leqslant\|\widehat{bx}\|_2=\|bx\|_2\leqslant\|x\|\,\|b\|_2=\|x\|\,\|\hat{b}\|_2.$$

Thus, there is $T\in\mathbb{B}(L^2(B))$ such that $T(\hat{b})=be_B(\hat{x})$. Since $T\in B'$, we get that $T\in JBJ$, which gives that $e_B(\hat{x})\in\hat{B}$. We therefore have a linear map $E_B:M\to B$ given by $\widehat{E_B(x)}=e_B(\hat{x})$. One checks that E satisfies all the conditions. ■

11. The hyperfinite II_1 factor

For $n \geq 1$, let $A_n = \mathbb{M}_{2^n}(\mathbb{C})$ and $\tau_n = \mathrm{Tr}/2^n : \mathbb{A}_n \to \mathbb{C}$ be the normalized trace. Consider the diagonal embedding $A_n \subset A_{n+1}$ given by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

Define $A = \cup_{n\geq 1} A_n$ and notice that A is a $*$-algebra which is equipped with a norm $\|\cdot\|$ which satisfies $\|x^*x\| = \|x\|^2$, for all $x \in A$. Moreover, $\tau : A \to \mathbb{C}$ defined by $\tau(x) = \tau_n(x)$, if $x \in A_n$, is a faithful tracial state functional which satisfies $|\tau(x)| \leq \|x\|$, for all $x \in A$.

We denote by H the closure of A w.r.t. to the norm $\|x\|_2 = \tau(x^*x)^{1/2}$, and consider the GNS $*$-homomorphism $\pi : A \to \mathbb{B}(H)$ given by $\pi(x)(\hat{y}) = \widehat{xy}$, for all $x, y \in A$.

Theorem 11.1. $R := \overline{\pi(A)}^{WOT}$ *is a II_1 factor and the map $\varphi : R \to \mathbb{C}$ given by $\varphi(x) = \langle x\hat{1}, \hat{1}\rangle$ is a normal faithful tracial state.*

Proof. Showing that φ is tracial on R is equivalent to proving that $\langle y\hat{1}, x^*\hat{1}\rangle = \langle x\hat{1}, y^*\hat{1}\rangle$, for all $x, y \in R$. Since this holds for all $x, y \in \pi(A)$ ($\langle \pi(y)\hat{1}, \pi(x)^*\hat{1}\rangle = \langle \hat{y}, \hat{x^*}\rangle = \tau(xy)$, for all $x, y \in A$) and $\pi(A)$ is SOT-dense in R, we deduce that φ is tracial on R.

Given $z \in A$, we have $\|yz\|_2^2 = \tau(z^*y^*yz) = \tau(yzz^*y^*) \leq \|zz^*\|\tau(yy^*) = \|z\|^2 \cdot \|y\|_2^2$, for all $y \in A$. This implies we the existence of an operator $\rho(z) \in \mathbb{B}(H)$ such that $\rho(z)(\hat{y}) = \widehat{yz}$, for all $y \in A$. Since $\rho(z) \in \pi(A)'$, we get that $\rho(z) \in R'$. Now, if $x \in R$ is such that $\varphi(x^*x) = 0$, then $x\hat{1} = 0$, and thus for every $z \in A$ we have that $x\hat{z} = x(\rho(z)\hat{1}) = \rho(z)(x\hat{1}) = 0$. Since $\hat{A}$ is dense in H, we conclude that $x = 0$, showing that φ is faithful on R. Since φ is clearly a normal state, the second assertion of the theorem is proven.

Finally, let us show that R is a factor. To this end, let $x \in \mathcal{Z}(R)$ and put $x_0 = x - \varphi(x) \cdot 1$. For $n \geq 1$, let $R_n = \pi(A_n) \subset R$ and $E_n : R \to R_n$ be the unique φ-preserving conditional expectation. Then $E_n(x) \in \mathcal{Z}(R_n)$. Since $R_n \cong \mathbb{M}_{2^n}(\mathbb{C})$ is factor and E_n is φ-preserving, we get that $E_n(x) = \varphi(E_n(x)) \cdot 1 = \varphi(x) \cdot 1$ or equivalently $E_n(x_0) = 0$. Thus, for every $n \geq 1$ and $y \in R_n$ we have that $\varphi(x_0y) = \varphi(E_n(x_0y)) = \varphi(E_n(x_0)y) = 0$. Hence, $\varphi(x_0y) = 0$, for all $y \in \pi(A)$. Since $\pi(A)$ is SOT-dense in R, we conclude that this equality holds for every $y \in R$. In particular, we have that $\varphi(x_0x_0^*) = 0$. Since φ is faithful we conclude that $x_0 = 0$ and thus $x = \varphi(x) \cdot 1 \in \mathbb{C} \cdot 1$. ■

Definition 11.2. A von Neumann algebra M is called **hyperfinite** if it admits an increasing sequence $(M_n)_{n\geq 1}$ of finite dimensional $*$-subalgebras such that $\cup_{n\geq 1} M_n$ is SOT-dense in M.

The II_1 factor R from Theorem 11.1 is hyperfinite by definition. Murray and von Neumann [MvN43] proved that any hyperfinite II_1 factor is isomorphic to R, which justifies the following:

Definition 11.3. The II_1 factor R is called *the* **hyperfinite II_1 factor.**

As it turns out, R is the smallest II_1 factor:

Exercise 11.4. Let M be a II_1 factor and $\tau : M \to \mathbb{C}$ be a faithful normal tracial state.

(1) Prove that there exists a projection $p \in M$ such that $\tau(p) = 1/2$.
(2) Prove that there exists an injective unital $*$-homomorphism $\rho : \mathbb{M}_2(\mathbb{C}) \to M$.
(3) Prove that there exists an injective unital $*$-homomorphism $\pi : R \to M$.

12. Group and group measure space von Neumann algebras

12.1. **Group von Neumann algebras.** Let Γ be a countable group. The left and right regular representations $\lambda, \rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ are given by $\lambda(g)(\delta_h) = \delta_{gh}$ and $\rho(g)(\delta_h) = \delta_{hg^{-1}}$. The **group von Neumann algebra** $L(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$ is the WOT-closure of the linear span of $\{\lambda(g)|g \in \Gamma\}$. We denote by $R(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$ the WOT-closure of the linear span of $\{\rho(g)|g \in \Gamma\}$.

Convention. Following the tradition in the subject, we denote $u_g = \lambda(g)$, for $g \in \Gamma$.

Proposition 12.1. *$\tau : L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful normal tracial state. Moreover, $L(\Gamma)' = R(\Gamma)$.*

Proof. Since $\tau(1) = 1$ and $\tau(x^*x) = \|x\delta_e\|^2 \geqslant 0$, for all $x \in M$, we get that τ is a normal state. Since $\tau(u_gu_h) = \tau(u_{gh}) = \delta_{gh,e} = \delta_{hg,e} = \tau(u_{hg}) = \tau(u_hu_g)$, we get that τ is a trace. If $\tau(x^*x) = 0$, then the first line of the proof implies that $x\delta_e = 0$. If $g \in \Gamma$, then $x\delta_g = x(\rho(g^{-1})\delta_e) = \rho(g^{-1})(x\delta_e) = 0$. This implies that $x = 0$, hence τ is faithful.

We identify $L^2(L(\Gamma))$ with $\ell^2\Gamma$ via the unitary $u_g \to \delta_g$. Under this identification, the involution J becomes $J(\delta_g) = \delta_{g^{-1}}$. Now, if $g, h \in \Gamma$, then $Ju_gJ(\delta_h) = Ju_g\delta_{h^{-1}} = J\delta_{gh^{-1}} = \delta_{hg^{-1}} = \rho(g)(\delta_h)$. This shows that $Ju_gJ = \rho(g)$, for all $g \in \Gamma$, hence $L(\Gamma)' = JL(\Gamma)J = R(G)$. ■

Notation 12.2. For $x \in L(\Gamma)$, we write $x\delta_e = \sum_{g\in\Gamma} x_g\delta_g \in \ell^2\Gamma$. Observe that in the above identification $L^2(L(\Gamma)) = \ell^2(\Gamma)$, we have that $\hat{x} = x\delta_e$. The coefficients $\{x_g\}_{g\in\Gamma}$ are called the **Fourier coefficients** of x and can be calculated as $x_g = \langle x\delta_e, \delta_g\rangle = \tau(xu_g^*)$. We will write $x = \sum_{g\in\Gamma} x_gu_g$, where the convergence holds in the $\|\cdot\|_2$ (but not necessarily the WOT).

Exercise 12.3. Let $x, y \in L(\Gamma)$ and let $x = \sum_{g\in\Gamma} x_gu_g, y = \sum_{g\in\Gamma} y_gu_g$ be their Fourier expansions. Prove that $x^* = \sum_{g\in\Gamma} \overline{x_{g^{-1}}}u_g$ and $xy = \sum_{g\in\Gamma}(\sum_{h\in\Gamma} x_hy_{h^{-1}g})u_g$.

By Proposition 12.1, $L(\Gamma)$ is a tracial von Neumann algebra. The next result clarifies when $L(\Gamma)$ is a II_1 factor.

Proposition 12.4. *Let Γ be a countable group. Then $L(\Gamma)$ is a factor if and only Γ has infinite conjugacy classes (or, is* **icc***): the conjugacy class $\{hgh^{-1}|h \in \Gamma\}$ is infinite, for every $g \in \Gamma \setminus \{e\}$.*

Proof. ($\Rightarrow$) Assume that $C = \{hgh^{-1}|h \in \Gamma\}$ is finite, for some $g \neq e$. Then $x = \sum_{k\in C} u_k$ belongs to the center of $L(\Gamma)$ and $x \notin \mathbb{C}\cdot 1$.

($\Leftarrow$) Assume that Γ is icc and let x be an element in the center of $L(\Gamma)$. Let $x = \sum_{g\in\Gamma} x_gu_g$ be the Fourier expansion of x and $h \in \Gamma$. Let $y = \sum_{g\in\Gamma} y_gu_g$ for the Fourier expansion of $y = u_hxu_h^*$. Then $y_g = \tau(yu_g^*) = \tau(u_hxu_h^*u_g^*) = \tau(xu_{hgh^{-1}}^*) = x_{hgh^{-1}}$. On the other hand, since x commutes with u_h, we get that $y = x$. Hence $x_{hgh^{-1}} = x_g$, for all $g, h \in \Gamma$. Since $\sum_{g\in\Gamma} |x_g|^2 < \infty$, and Γ is icc, we conclude that $x_g = 0$, for all $g \in \Gamma \setminus \{e\}$. Thus, $x \in \mathbb{C}\cdot 1$. ■

Exercise 12.5. Prove that the following groups are icc:

(1) the group S_∞ of bijections $\pi : \mathbb{N} \to \mathbb{N}$ such that $\{n \in \mathbb{N} \mid \pi(n) \neq n\}$ is finite.
(2) the free product group $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1, Γ_2 are any group with $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$. (in particular, the free group $\mathbb{F}_n$ on $n \geq 2$ generators is icc).
(3) $\mathrm{SL}_n(\mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z}) \mid \det(A) = 1\}$, for every odd $n \geqslant 3$.

12.2. **Group measure space von Neumann algebras.** Let Γ be a countable group and (X,μ) a standard probability space. We say that an action $\Gamma \curvearrowright (X,\mu)$ is **probability measure preserving** (abbreviated, **pmp**) if for every $g\in\Gamma$ the map $X\ni x\mapsto g\cdot x\in X$ is measurable and measure preserving: $\mu(g\cdot Y)=\mu(Y)$, for every measurable set $Y\subset X$.

Define a unitary representation $\sigma:\Gamma\to\mathcal{U}(L^2(X))$ by $\sigma_g(f)(x)=f(g^{-1}x)$, for all $f\in L^2(X)$. Note that $\sigma_g(L^\infty(X))=L^\infty(X)$, for all $g\in\Gamma$. Further, we denote $H=L^2(X)\otimes\ell^2\Gamma$ and define a unitary representation $u:\Gamma\to\mathcal{U}(H)$ be letting $u_g=\sigma_g\otimes\lambda(g)$. We also define a $*$-homomorphism $\pi:L^\infty(X)\to\mathbb{B}(H)$ be letting $\pi(f)(\xi\otimes\delta_g)=f\xi\otimes\delta_g$, and view $L^\infty(X)\subset\mathbb{B}(H)$, via π. Then

$$u_gfu_g^*=\sigma_g(f),\text{ for all } g\in\Gamma \text{ and every } f\in L^\infty(X).$$

Definition 12.6. The **group measure space von Neumann algebra** $L^\infty(X)\rtimes\Gamma\subset\mathbb{B}(H)$ is defined as the WOT-closure of the linear span of $\{fu_g|f\in L^\infty(X),g\in\Gamma\}$.

Proposition 12.7. *$\tau:L^\infty(X)\rtimes\Gamma\to\mathbb{C}$ given by $\tau(x)=\langle x(1\otimes\delta_e),1\otimes\delta_e\rangle$ is a faithful normal tracial state.*

Proof. Note that for all $f\in L^\infty(X)$ and $g\in\Gamma$ we have that

$$\tau(fu_g)=\langle fu_g(1\otimes\delta_e),1\otimes\delta_e\rangle=\langle f\otimes\delta_g,1\otimes\delta_e\rangle=\delta_{g,e}\int_X f\,\mathrm{d}\mu.$$

If $f_1,f_2\in L^\infty(X)$ and $g_1,g_2\in\Gamma$, then $f_1u_{g_1}f_2u_{g_2}=f_1\sigma_{g_1}(f_2)u_{g_1g_2}$ and $f_2u_{g_2}f_1u_{g_2}=f_2\sigma_{g_2}(f_1)u_{g_2g_1}$. Since $\tau(\sigma_g(f))=\tau(f)$, for all $f\in L^\infty(X)$ and $g\in\Gamma$, we get that $\tau(f_1u_{g_1}f_2u_{g_2})=\tau(f_2u_{g_2}f_1u_{g_2})$. This implies that τ is a trace. We leave the rest of the proof as an exercise. ■

Proposition 12.8. *Every $a\in M$ has a unique* **Fourier expansion** *of the form $a=\sum_{g\in\Gamma}a_gu_g$, where $a_g=E_A(au_g^*)\in A$, where the series converges in $\|\cdot\|_2$. Moreover, we have the following:*

- $a^*=\sum_{g\in\Gamma}\sigma_{g^{-1}}(a_g^*)u_g$.
- $\|a\|_2^2=\sum_{g\in\Gamma}\|a_g\|_2^2$.
- $ab=\sum_{g\in\Gamma}(\sum_{h\in\Gamma}a_h\sigma_h(b_{h^{-1}g}))u_g$.

Proof. The formula $U(fu_g)=f\otimes\delta_g$ defines a unitary operator $U:L^2(M)\to L^2(X)\otimes\ell^2\Gamma$. Thus, every $a\in M$ can be written as $a=\sum_{g\in\Gamma}a_gu_g$, where $a_g\in L^2(X)$ satisfy $\sum_{g\in\Gamma}\|a_g\|_2^2=\|a\|_2^2$. Moreover, we have that $\hat{a}_e=e_A(\hat{a})$ and thus $a_e=E_A(a)$. Since $au_h^*=\sum_{g\in\Gamma}a_{gh}u_g$, we get that $a_h=E_A(au_h^*)$, for every $h\in\Gamma$. We leave the rest of the proof as an exercise. ■

Definition 12.9. A pmp action $\Gamma\curvearrowright(X,\mu)$ is called:

- **ergodic** if every Γ-invariant measurable set $Y\subset X$ satisfies $\mu(Y)\in\{0,1\}$.
- **(essentially) free** if $\mu(\{x\in X\mid gx=x\})=0$, for every $g\in\Gamma\setminus\{e\}$.

Lemma 12.10. *A pmp action $\Gamma\curvearrowright(X,\mu)$ is ergodic if and only if any function $f\in L^2(X)$ which satisfies that $\sigma_g(f)=f$, for all $g\in\Gamma$, is essentially constant.*

Proof. ($\Leftarrow$) If Y is a Γ-invariant set, then $f=1_Y\in L^2(X)$ is a Γ-invariant function. Thus, there is $c\in\mathbb{C}$ such that $f=c$. As $f^2=f$, we get that $c\in\{0,1\}$, hence $\mu(Y)=\int_X f\,\mathrm{d}\mu=c\in\{0,1\}$.

($\Rightarrow$) Let $f\in L^2(X)$ be a Γ-invariant function. If f is not constant, then it admits at least two distinct essential values $z,w\in\mathbb{C}$. Let $\delta=|z-w|/2$. Then $Y=\{x\in X|\ |f(x)-z|<\delta\}$ and $Z=\{x\in X|\ |f(x)-w|<\delta\}$ are disjoint, Γ-invariant, measurable sets. Since $\mu(Y)>0$ and $\mu(Z)>0$, we get a contradiction with the ergodicity of the action. ■

Exercise 12.11. Let Γ be an infinite group and (Y,ν) be a non-trivial standard probability space. Define $(X,\mu)=(Y^\Gamma,\nu^{\otimes\Gamma})$ Consider the **Bernoulli action** $\Gamma\curvearrowright(X,\mu)$ given by

$$g\cdot x=(x_{g^{-1}h})_{h\in\Gamma},\text{ for every } g\in\Gamma \text{ and } x=(x_h)_{h\in\Gamma}\in X.$$

Prove that this action is pmp, essentially free and ergodic. Moreover, prove that this action is **mixing**: if $Y,Z\subset X$ are measurable sets, then $\lim\limits_{g\to\infty}\mu(gY\cap Z)=\mu(Y)\mu(Z)$.

Exercise 12.12. Let G be a compact group and $\Gamma<G$ be a countable dense subgroup (e.g., take $G=\mathbb{T}$ and $\Gamma=\{\exp(2\pi i\ n\alpha)|n\in\mathbb{Z}\}$, where $\alpha\in\mathbb{R}\setminus\mathbb{Q}$). Let m_G be the Haar measure of G. Consider the **left translation action** $\Gamma\curvearrowright(G,m_G)$ given by left multiplication: $g\cdot x=gx$. Prove that this action is pmp, essentially free and ergodic. Prove that this action is not mixing.

Proposition 12.13. *Let $\Gamma\curvearrowright(X,\mu)$ be a pmp action. Denote $M=L^\infty(X)\rtimes\Gamma$ and $A=L^\infty(X)$.*

(1) The action $\Gamma\curvearrowright(X,\mu)$ is free if and only if $A\subset M$ is maximal abelian, i.e. $A'\cap M=A$.
(2) Assume that the action $\Gamma\curvearrowright(X,\mu)$ is free. Then M is a factor if and only if the action $\Gamma\curvearrowright(X,\mu)$ is ergodic.

Proof. (1) Assume that $A'\cap M=A$. Let $g\in\Gamma\setminus\{e\}$ and put $Y=\{x\in X|gx=x\}$. Since $1_Y\sigma_g(f)=1_Yf$, for all $f\in A$, we get that $a=1_Yu_g\in A'\cap M$. Hence $a\in A$ and thus $a=E_A(a)=0$, showing that $\mu(Y)=0$. This shows that the action is free.

Conversely, assume that the action is free. Let $a\in A'\cap M$ and $a=\sum_{g\in\Gamma}a_gu_g$ be its Fourier decomposition. If $b\in A$, then $\sum_{g\in\Gamma}ba_gu_g=ba=ab=\sum_{g\in\Gamma}a_g\sigma_g(b)u_g$, thus $ba_g=\sigma_g(b)a_g$, for all $g\in\Gamma$. Let $g\in\Gamma\setminus\{e\}$ and put $Y_g=\{x\in X|a_g(x)\neq 0\}$. From the last equality we get that $b(g^{-1}x)=b(x)$, for almost every $x\in Y_g$. Since (X,μ) is a standard probability space, we can find a sequence of measurable sets $X_n\subset X$, $n\geq 1$, which separate points in X. By applying the last identity to $b=1_{X_n}$, for all $n\geq 1$, we deduce that $g^{-1}x=x$, for almost every $x\in Y_g$. Since the action is free, we get that $\mu(Y_g)=0$, hence $a_g=0$. Since this holds for all $g\in\Gamma\setminus\{e\}$, we conclude that $a\in A$.

(2) Since the action is free, (1) implies that $\mathcal{Z}(M)=A\cap M'=\{a\in A\mid\sigma_g(a)=a,\forall g\in\Gamma\}$. By Lemma 10.8, the conclusion follows. ∎

Exercise 12.14. Let Γ be an icc group and $\Gamma\curvearrowright(X,\mu)$ be a pmp action. Prove that $L^\infty(X)\rtimes\Gamma$ is a II_1 factor if and only if the action $\Gamma\curvearrowright(X,\mu)$ is ergodic.

12.3. **Cartan subalgebras and orbit equivalence.**

Definition 12.15. Let (M,τ) be a tracial von Neumann algebra. We say that a von Neumann subalgebra $A\subset M$ is a **Cartan subalgebra** if the following conditions are satisfied:

(1) A is maximal abelian, i.e., $A'\cap M=A$, and
(2) the normalising group $\mathcal{N}_M(A)=\{u\in\mathcal{U}(M)\mid uAu^*=A\}$ satisfies $\mathcal{N}_M(A)''=M$.

By Proposition 12.13, $L^\infty(X)\subset L^\infty(X)\rtimes\Gamma$ is a Cartan subalgebra, for every free pmp action $\Gamma\curvearrowright(X,\mu)$. It is a fundamental observation of Singer (1955) that the isomorphism class of the inclusion $L^\infty(X)\subset L^\infty(X)\rtimes\Gamma$ captures exactly the orbit equivalence class of the action $\Gamma\curvearrowright(X,\mu)$. An **isomorphism** between two standard probability spaces (X,μ) and (Y,ν) is a Borel isomorphism $\theta:X'\to Y'$ between Borel co-null subsets $X'\subset X,Y'\subset Y$ which preserves the measure, i.e., satisfies $\mu(\theta^{-1}(Z))=\nu(Z)$, for all measurable subsets $Z\subset Y'$.

Proposition 12.16. *If $\Gamma\curvearrowright(X,\mu)$ and $\Lambda\curvearrowright(Y,\nu)$ are free pmp actions, then the following conditions are equivalent:*

(1) *There exists a $*$-isomorphism $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ such that $\pi(L^\infty(X)) = L^\infty(Y)$.*
(2) *The actions are* **orbit equivalent**, *i.e., there exists an isomorphism $\theta : (X, \mu) \to (Y, \nu)$ which takes the Γ-orbits onto the Λ-orbits: $\theta(\Gamma \cdot x) = \Lambda \cdot \theta(x)$, for almost every $x \in X$. (In this case, θ is called an orbit equivalence between the actions.)*

Proof. Denote $A = L^\infty(X), B = L^\infty(Y), M = L^\infty(X) \rtimes \Gamma$ and $N = L^\infty(Y) \rtimes \Lambda$.

(1) $\Rightarrow$ (2) Since $\pi_{|A} : A \to B$ is a $*$-isomorphism, we can find an isomorphism $\theta : (X, \mu) \to (Y, \nu)$ such that $\pi(a) = a \circ \theta^{-1}$, for all $a \in A$ (see, e.g., [AP18, Theorem 3.3.4]). We will prove that θ is the desired orbit equivalence between the actions.

To this end, fix $g \in \Gamma$ and denote $v = \pi(u_g)$. Then v normalises B and thus we can find an isomorphism $\alpha : (Y, \nu) \to (Y, \nu)$ such that $vbv^* = b \circ \alpha$, for all $b \in B$.

Claim. $\alpha(y) \in \Lambda \cdot y$, for almost every $y \in Y$.

To this end, consider the Fourier expansion $v = \sum_{h \in \Lambda} v_h u_h$, where $v_h \in B$ for all $h \in \Lambda$. Since $vb = (b \circ \alpha)$, for all $b \in B$, we deduce that $v_h(b \circ h^{-1}) = v_h(b \circ \alpha)$, for all $h \in \Lambda$ and $b \in B$. If we let $Y_h = \{y \in Y \mid v_h(y) \neq 0\}$, then the same argument as in the proof of Proposition 12.13 shows that $\alpha(y) = h^{-1} \cdot y$, for almost every $y \in Y_h$ and all $h \in \Lambda$. Now, if we let $Z = Y \setminus (\cup_{h \in \Lambda} Y_h)$, then $1_Z v_h = 0$, for all $h \in \Lambda$ and thus $1_Z v = \sum_{h \in \Lambda}(1_Z v_h) u_h = 0$. Hence $\nu(Z)^{1/2} = \|1_Z\|_2 = \|1_Z v\|_2 = 0$, which implies that the set $\cup_{h \in \Lambda} Y_h$ is co-null in Y. This clearly implies the claim.

Finally, if $a \in A$, then $a \circ g^{-1} \circ \theta^{-1} = \pi(u_g a u_g^*) = v\pi(a)v^* = \pi(a) \circ \alpha = a \circ \theta^{-1} \circ \alpha$. Thus, $g^{-1} \circ \theta^{-1} = \theta^{-1} \circ \alpha$ hence $\theta \circ g^{-1} = \alpha \circ \theta$. Together with the claim, this implies that for almost every $x \in X$, we have that $\theta(g^{-1} \cdot x) = \alpha(\theta(x)) \in \Lambda \cdot \theta(x)$. Since $g \in \Gamma$ is arbitrary, we conclude that $\theta(\Gamma \cdot x) \subset \Lambda \cdot \theta(x)$, for almost every $x \in X$. Since the reverse inclusion can be proved similarly, it follows that θ is an orbit equivalence.

(2) $\Rightarrow$ (1) Let $\theta : (X, \mu) \to (Y, \nu)$ be an orbit equivalence. Define a $*$-isomorphism $\pi : A \to B$ by letting $\pi(a) = a \circ \theta^{-1}$. Our goal is to show that π extends to a $*$-isomorphism $\pi : M \to N$.

To this end, fix $g \in \Gamma$. Then $(\theta \circ g^{-1} \circ \theta^{-1})(y) \in \Lambda \cdot y$, for almost every $y \in Y$. For $h \in \Lambda$, put $Y_{g,h} = \{y \in Y \mid (\theta \circ g^{-1} \circ \theta^{-1})(y) = h^{-1} \cdot y\}$. Then $\{Y_{g,h}\}_{h \in \Lambda}$ is a measurable partition of Y. Since $h^{-1} \cdot Y_{g,h} = \{y \in Y \mid (\theta \circ g \circ \theta^{-1})(y) = h \cdot y\}$, we also have that $\{h^{-1} \cdot Y_{g,h}\}_{h \in \Lambda}$ is a measurable partition of Y. Using the last two facts, one checks that the formula $\pi(u_g) = \sum_{h \in \Lambda} 1_{A_h} u_h$ defines a unitary in N such that $\pi(u_g)\pi(a)\pi(u_g)^* = \pi(a \circ g^{-1})$, for all $a \in A$. This entails that π extends to a $*$-homomorphism from the linear span of $\{au_g \mid a \in A, g \in \Gamma\}$ to N. Moreover, π is trace preserving. We leave it as an exercise to show that π extends to a $*$-isomorphism $\pi : M \to N$. ■

13. Amenable groups and von Neumann algebras

13.1. Amenable groups.

Definition 13.1. A countable group Γ is called **amenable** if there exists a state $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ which is invariant under the left translation action: $\varphi(g \cdot f) = \varphi(f)$, for all $g \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Here, $g \cdot f \in \ell^\infty(\Gamma)$ is defined as $(g \cdot f)(h) = f(g^{-1}h)$.

Example 13.2. Every finite group Γ is amenable, as witnessed by the state $\varphi(f) = |\Gamma|^{-1} \sum_{g \in \Gamma} f(g)$.

In order to give examples of infinite amenable groups, we need to recall the following:

Definition 13.3. The **Stone-Čech compactification** of $\mathbb{N}$ is defined as the maximal ideal space of the abelian C*-algebra $\ell^\infty(\mathbb{N})$ and is denoted by $\beta\mathbb{N}$. An **ultrafilter** of $\mathbb{N}$ is an element of $\beta\mathbb{N}$, i.e., a non-zero homomorphism $\omega : \ell^\infty(\mathbb{N}) \to \mathbb{C}$. For every $n \in \mathbb{N}$, we denote by $e_n \in \beta\mathbb{N}$ the **evaluation** at n, i.e., $e_n(f) = f(n)$. An ultrafilter $\omega \in \beta\mathbb{N}$ is **free** if it does not belong to $\mathbb{N} \equiv \{e_n\}_{n \in \mathbb{N}}$.

Notation. If $\omega \in \beta\mathbb{N}$, we denote $\lim_{n\to\omega} x_n := \omega((x_n)_n)$, for every $(x_n)_n \in \ell^\infty(\mathbb{N})$.

Remark 13.4. We have that $\beta\mathbb{N} \setminus \mathbb{N} \neq \emptyset$. To see this, let $K_n \subset \beta\mathbb{N}$ be the weak*-closure of $\{e_k \mid k > n\}$. Then K_n is weak*-compact by Alaoglu's theorem and $K_{n+1} \subset K_n$, for all n. Thus, $\cap_n K_n \neq \emptyset$. If $\omega \in \cap_n K_n$, then $\omega \in K_n$ and thus $\omega(\delta_n) = 0$, for all $n \in \mathbb{N}$. This shows that $\omega \notin \mathbb{N}$.

Exercise 13.5. If $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\omega} x_n = 0$.

Examples 13.6. (of amenable groups) Let ω be a free ultrafilter on $\mathbb{N}$.

(1) Assume that $\Gamma = \cup_n \Gamma_n$, where $\Gamma_n < \Gamma$ are amenable subgroups and $\Gamma_n \subset \Gamma_{n+1}$, for all n. Then Γ is amenable. To see this, let $\varphi_n : \ell^\infty(\Gamma_n) \to \mathbb{C}$ be a left invariant state. Define $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ by $\varphi(f) = \lim_{n\to\omega} \varphi_n(f_{|\Gamma_n})$. Then φ is a left invariant state. Indeed, if $g \in \Gamma$, then $g \in \Gamma_N$, for some N. Thus, if $f \in \ell^\infty(\Gamma)$, then $(g\cdot f)_{|\Gamma_n} = g\cdot(f_{|\Gamma_n})$, for all $n \geq N$. Hence by Exercise 13.5 we get $\varphi(g \cdot f) = \lim_{n\to\omega} \varphi_n(g \cdot (f_{\Gamma_n})) = \lim_{n\to\omega} \varphi_n(f_{|\Gamma_n}) = \varphi(f)$. In particular, any increasing union of finite groups, such as S_∞, is amenable.

(2) $\mathbb{Z}$ is amenable. To see this, let $F_n = \{-n, -n+1, ..., -1, 0, 1, ..., n-1, n\}$. Then for any $g \in \mathbb{Z}$ we have $|(g + F_n) \setminus F_n| \leq 2|g|$ and thus $\lim_{n\to\infty} |(g + F_n) \setminus F_n|/|F_n| = 0$. Define $\varphi : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ by letting $\varphi(f) = \lim_{n\to\omega}(1/|F_n|)\sum_{x\in F_n} f(x)$. For every $g \in \mathbb{Z}$, we have

$$\begin{aligned}|\varphi(g \cdot f) - \varphi(f)| &= |\lim_{n\to\omega}(1/|F_n|)(\sum_{x\in F_n} f(x-g) - \sum_{x\in F_n} f(x))| \\ &\leq \|f\|_\infty \lim_{n\to\omega} |(F_n - g)\triangle F_n|/|F_n| = 0.\end{aligned}$$

Theorem 13.7. *Let Γ be a countable group. Then the following conditions are equivalent:*

(1) Γ is amenable.

(2) Γ satisfies the **Reiter condition***: there exists a sequence of non-negative functions $f_n \in \ell^1(\Gamma)$ such that $\|f_n\|_1 = 1$, for all n, and $\lim_{n\to\infty} \|f_n \circ g - f_n\|_1 = 0$, for all $g \in \Gamma$.*

(3) Γ satisfies the **Følner condition***: there exists a sequence of finite subsets $F_n \subset \Gamma$ such that $\lim_{n\to\infty} |gF_n \setminus F_n|/|F_n| = 0$, for all $g \in \Gamma$.*

(4) the left regular representation of Γ has **almost invariant vectors***: there exists a sequence $\xi_n \in \ell^2(\Gamma)$ such that $\|\xi_n\|_2 = 1$, for all n, and $\lim_{n\to\infty} \|\lambda(g)\xi_n - \xi_n\|_2 = 0$, for all $g \in \Gamma$.*

The proof of this result relies on two very useful tricks, due to Day (the proof of (1) $\Rightarrow$ (2)) and Namioka (the proof of (2) $\Rightarrow$ (3)).

Proof. Enumerate $\Gamma = \{g_n\}_{n\geq 1}$.

(1) $\Rightarrow$ (2) Fix $n \geq 1$ and consider the convex subset

$$C := \{(f \circ g_1 - f, f \circ g_2 - f, ..., f \circ g_n - f) \mid f \in \ell^1(\Gamma), f \geq 0\}$$

of the Banach space $\ell^1(\Gamma)^{\oplus n}$ with the norm $\|(f_1, f_2, ..., f_n)\| = \sum_{i=1}^n \|f_i\|_1$.

We claim that $\mathbf{0} = (0, 0, ..., 0) \in \overline{C}^{\|\cdot\|}$. Assuming this claim, we can find $f_n \in \ell^1(\Gamma)$ such that $f_n \geq 0, \|f_n\|_1 = 1$ and $\sum_{i=1}^n \|f_n \circ g_i - f_n\|_1 \leq 1/n$. This clearly implies (2).

If the claim were false, then since $\overline{C}^{\|\cdot\|} \subset \ell^1(\Gamma)^{\oplus n}$ is a closed convex set and $(\ell^1(\Gamma)^{\oplus n})^* = \ell^\infty(\Gamma)^{\oplus n}$, the Hahn-Banach separation theorem implies the existence of $F_1, F_2, ..., F_n \in \ell^\infty(\Gamma)$ and $\alpha > 0$ such that $\sum_{i=1}^n \Re\langle f \circ g_i - f, F_i\rangle \geq \alpha$, for any $f \in \ell^1(\Gamma)$ with $f \geq 0$ and $\|f\|_1 = 1$.

If we put $F = \sum_{i=1}^n \Re(F_i \circ g_i^{-1} - F_i)$, then the last inequality rewrites as $\langle f, F\rangle \geq \alpha$, for any $f \in \ell^1(\Gamma)$ with $f \geq 0$ and $\|f\|_1 = 1$. For $f = \delta_g$, this implies that $F(g) \geq \alpha$, for all $g \in \Gamma$. Thus, we get that $\varphi(F) \geq \varphi(\alpha \cdot 1) = \alpha > 0$. On the other hand, $\varphi(F) = \sum_{i=1}^n(\varphi(\Re F_i \circ g_i^{-1}) - \varphi(\Re F_i)) = 0$. This gives the desired contradiction.

(2) $\Rightarrow$ (3) If $f_1, f_2 \in \ell^1(\Gamma)$ and $f_1, f_2 \geq 0$, then Fubini's theorem implies that

$$\|f_1 - f_2\|_1 = \int_0^\infty \|1_{\{f_1>t\}} - 1_{\{f_2>t\}}\|_1 \mathrm{d}t \quad \text{and} \quad \|f_1\|_1 = \int_0^\infty \|1_{\{f_1>t\}}\mathrm{d}t. \tag{13.1}$$

By (2), for any $n \geq 1$ we can find $f \in \ell^1(\Gamma)$ such that $f \geq 0, \|f\|_1 = 1$ and $\sum_{i=1}^n \|f \circ g_i - f\|_1 < 1/n$. For $t > 0$, let $K_t = \{f > t\}$. Since $f \in \ell^1(\Gamma)$, we get that K_t is a finite subset of Γ. Also, note that $\{f \circ g > t\} = g^{-1}K_t$ and thus that $\|1_{\{f\circ g>t\}} - 1_{\{f>t\}}\|_1 = |g^{-1}K_t \triangle K_t|$, for all $g \in \Gamma$. Thus, by combining the last inequality with (13.1), we derive that

$$\int_0^\infty \sum_{i=1}^n |g_i^{-1}K_t - K_t|\mathrm{d}t < 1/n = 1/n\|f\|_1 = \int_0^\infty (|K_t|/n)\mathrm{d}t.$$

Hence, there is $t_n > 0$ such that $F_n := K_{t_n}$ satisfies $\sum_{i=1}^n |g_i^{-1}F_n \triangle F_n| < |F_n|/n$. This proves (3).

(3) $\Rightarrow$ (4) Let $\xi_n := 1_{F_n}/\sqrt{|F_n|}$. Then $\|\xi_n\|_2 = 1$ and $\|\lambda(g)\xi_n - \xi_n\|_2 = \sqrt{|gF_n \triangle F_n|/|F_n|}$, for all $n \geq 1$ and $g \in \Gamma$, which clearly implies (4).

(4) $\Rightarrow$ (1) Let ω be a free ultrafilter on $\mathbb{N}$. Define $\varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ by letting $\varphi(f) = \lim_{n\to\omega}\langle f \cdot \xi_n, \xi_n\rangle$. Then φ is a state and $\varphi(f \circ g) = \lim_{n\to\omega}\langle f(\xi_n \circ g^{-1}), \xi_n \circ g^{-1}\rangle = \varphi(f)$, for all $f \in \ell^1(\Gamma)$, $g \in \Gamma$. ∎

Proposition 13.8. $\mathbb{F}_2$ *is not amenable.*

Proof. Assume by contradiction that there exists a left translation invariant state $\varphi : \ell^\infty(\mathbb{F}_2) \to \mathbb{C}$. Define $m : \mathcal{P}(\mathbb{F}_2) \to [0, 1]$ my $m(A) = \varphi(1_A)$. Then m is finitely additive ($m(A\cup B) = m(A)+m(B)$, for every disjoint $A, B \subset \mathbb{F}_2$) and left invariant ($m(gA) = m(A)$, for every $g \in \mathbb{F}_2$ and $A \subset \mathbb{F}_2$).

Let a and b be the free generators of $\mathbb{F}_2$. Let S be the set of elements of $\mathbb{F}_2$ whose reduced form begins with a non-zero power of a, and put $T = \mathbb{F}_2 \setminus S$. Then $aT \subset S$, $bS\cup b^2S \subset T$ and $bS\cap b^2S = \emptyset$. Thus, we get $m(S) \geq m(aT) = m(T) \geq m(bS \cup b^2S) = m(bS) + m(b^2S) = 2m(S)$. This implies that $m(S) = m(T) = 0$. Since $m(S) + m(T) = m(\mathbb{F}_2) = 1$, this provides a contradiction. ∎

Exercise 13.9. Let Γ_1 and Γ_2 be any countable groups such that $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$. Prove that the free product group $\Gamma = \Gamma_1 * \Gamma_2$ is not amenable.

13.2. **Amenable von Neumann algebras.**

Definition 13.10. A tracial von Neumann algebra (M, τ) is called **amenable** if there exists a state $\Phi : \mathbb{B}(L^2(M)) \to \mathbb{C}$ such that $\Phi_{|M} = \tau$ and $\Phi(Tx) = \Phi(xT)$, for all $x \in M$ and $T \in \mathbb{B}(L^2(M))$.

Theorem 13.11. *Let Γ be a countable group. Then Γ is amenable if and only if $L(\Gamma)$ is amenable.*

Proof. As before, we identify $L^2(L(\Gamma)) \equiv \ell^2(\Gamma)$, in the natural way.

Assume that Γ is amenable and let $\varphi : \ell^\infty(\Gamma)$ be a left translation invariant state. Define a state $\Phi : \mathbb{B}(\ell^2(\Gamma)) \to \mathbb{C}$ by letting $\Phi(T) := \varphi(g \mapsto \langle T\delta_g, \delta_g\rangle)$. If $T \in L(\Gamma)$, then for all $g \in \Gamma$ we have

$$\langle T\delta_g, \delta_g\rangle = \langle T\rho(g)\delta_e, \rho(g)\delta_e\rangle = \langle \rho(g)^*T\rho(g)\delta_e, \delta_e\rangle = \langle T\delta_e, \delta_e\rangle = \tau(T),$$

and thus $\Phi(T) = \tau(T)$. If $T \in \mathbb{B}(\ell^2(\Gamma))$ and $h \in \Gamma$, then

$$\Phi(\lambda(h)T\lambda(h)^*) = \varphi(g \mapsto \langle\lambda(h)T\lambda(h)^*\delta_g, \delta_g\rangle) = \varphi(g \mapsto \langle T\delta_{h^{-1}g}, \delta_{h^{-1}g}\rangle) = \Phi(T).$$

Thus, if $\mathcal{C} := \{x \in L(\Gamma) \mid \Phi(Tx) = \Phi(xT)$, for all $T \in \mathbb{B}(\ell^2(\Gamma))\}$, then $\lambda(g) \in \mathcal{C}$, for all $g \in \Gamma$. By Cauchy-Schwarz, we have that $|\Phi(Tx)|^2 \leq \Phi(TT^*)\Phi(x^*x) \leq \|T\|^2\Phi(x^*x) = \|T\|^2\|x\|_2^2$ and similarly $|\Phi(xT)|^2 \leq \|T\|^2\|x\|_2^2$, for all $x \in L(\Gamma)$ and $T \in \mathbb{B}(\ell^2(\Gamma))$. This implies that $\mathcal{C}$ is $\|\cdot\|_2$-closed. Since $\mathcal{C}$ contains the linear span of $\lambda(\Gamma)$, we conclude that $\mathcal{C} = L(\Gamma)$. This shows that $L(\Gamma)$ is amenable.

Conversely, assume that $L(\Gamma)$ is amenable. Let Φ be a state on $\mathbb{B}(\ell^2(\Gamma))$ such that $\Phi(Tx) = \Phi(xT)$, for all $x \in L(\Gamma)$ and $T \in \mathbb{B}(\ell^2(\Gamma))$. Consider the natural embedding $\ell^\infty(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$ and notice that $\lambda(g)f\lambda(g)^* = f \circ g^{-1} = g \cdot f$, for all $f \in \ell^\infty(\Gamma)$ and $g \in \Gamma$. Thus, for all $f \in \ell^\infty(\Gamma)$ and $g \in \Gamma$, we have that $\Phi(g \cdot f) = \Phi(\lambda(g)f\lambda(g)^*) = \Phi(f)$. This implies that Γ is amenable. ■

Proposition 13.12. *Let (M, τ) be a hyperfinite tracial von Neumann algebra. Then M is amenable.*

Proof. Let (M_n) be an increasing sequence of finite dimensional $*$-subalgebras such that $\cup_n M_n$ is SOT-dense in M. Let ω be a free ultrafilter on $\mathbb{N}$ and denote by μ_n the Haar probability measure of the compact group $\mathcal{U}(M_n)$, for every $n \geq 1$. Let $\Phi : \mathbb{B}(L^2(M)) \to \mathbb{C}$ be given by

$$\Phi(T) = \lim_{n \to \omega} \int_{\mathcal{U}(M_n)} \langle T\hat{u}, \hat{u} \rangle \, \mathrm{d}\mu_n(u).$$

Then it is clear that $\Phi_{|M} = \tau$ and $\Phi(uTu^*) = \Phi(T)$, for all $u \in \cup_n \mathcal{U}(M_n)$ and $T \in \mathbb{B}(L^2(M))$. By reasoning as in the proof of Theorem 13.11, it follows that M is amenable. ■

13.3. **Connes' theorem.** In the early 1980s, Connes discovered that Hilbert bimodules provide an appropriate representation theory for tracial von Neumann algebras, paralleling the theory of unitary representations for groups.

Definition 13.13. Let (M, τ) be a tracial von Neumann algebra. A **Hilbert M-bimodule** is a Hilbert space H equipped with commuting normal $*$-homomorphisms $\pi : M \to \mathbb{B}(H)$, $\rho : M^{\mathrm{op}} \to \mathbb{B}(H)$, where M^{op} is the opposite von Neumann algebra of M. We write $x\xi y = \pi(x)\rho(y^{\mathrm{op}})\xi$.

Examples 13.14. (of Hilbert bimodules)

1. The **trivial bimodule** $L^2(M)$.
2. The **coarse bimodule** $L^2(M) \bar{\otimes} L^2(M)$ with $x(\xi \otimes \eta)y = x\xi \otimes \eta y$.

Theorem 13.15 (Connes, 1975)**.** *Let (M, τ) be a tracial von Neumann algebra. Assume that M is separable, i.e., $L^2(M)$ is separable. Then the following are equivalent:*

(1) M is amenable.
(2) M is **injective***: there exists a conditional expectation $E : \mathbb{B}(L^2(M)) \to M$.*
(3) the coarse M-bimodule has **almost central vectors***: there exists a sequence $\xi_n \in L^2(M) \bar{\otimes} L^2(M)$ such that $\langle x\xi_n, \xi_n \rangle = \tau(x)$ for all $n \geq 1$ and $\lim_{n \to \infty} \|x\xi_n - \xi_n x\| = 0$, for all $x \in M$.*
(4) M is hyperfinite.

In particular, any separable amenable II_1 factor M is isomorphic to R. Thus, any II_1 factor of the form $L(\Gamma)$ or $L^\infty(X) \rtimes \Gamma$, with Γ amenable, is isomorphic to R.

In the case of group measure space algebras associated to free pmp actions, Connes' result was strengthened as follows:

Theorem 13.16 (Ornstein-Weiss, 1980)**.** *Let Γ, Λ be amenable groups. Then any ergodic pmp actions $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent.*

In the case $\Gamma = \Lambda$ this result was first established by Dye (1959). This result was generalized to amenable countable pmp equivalence relations by Connes-Feldman-Weiss (1981).

14. Non-orbit equivalent actions

Theorem 14.1 (Gaboriau, 1998). *If $2 \leq m \neq n \leq \infty$, then any two free ergodic pmp actions $\mathbb{F}_m \curvearrowright (X, \mu)$ and $\mathbb{F}_n \curvearrowright (Y, \nu)$ are* **not** *orbit equivalent.*

We reproduce a proof of this theorem presented in [AP18, Section 18.3] and due to Vaes, itself a version in the spirit of operator algebras of a previous proof by Gaboriau. This is based on the following result showing that the **cost** (an orbit equivalence invariant) of any free ergodic pmp action of $\mathbb{F}_m$ is exactly m.

Theorem 14.2 (Gaboriau, 1998). *Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic pmp action of $\Gamma = \mathbb{F}_m$. Denote by $\mathcal{R} := \{(x, g \cdot x) \mid x \in X, g \in \Gamma\}$ the orbit equivalence relation. Let $(g_k)_k \subset \Gamma$ be group elements and $(A_k)_k \subset X$ be measurable sets such that $\mathcal{R}$ is the smallest equivalence relation on X containing $\cup_k \{(x, g_k \cdot x) \mid x \in A_k\}$. Then $\sum_k \mu(A_k) \geq m$.*

Proof of Theorem 14.2. Denote $M = L^\infty(X) \rtimes \Gamma$ and let $(u_g)_{g \in \Gamma} \subset \mathcal{U}(M)$ be the canonical unitaries. Let $a_1, ..., a_m \in \Gamma$ be free generators. We consider the space of 1-cocycles

$$Z^1(\Gamma, M) = \{c : \Gamma \to M \mid c(gh) = u_g c(h) + c(g), \text{ for all } g, h \in \Gamma\}.$$

Notice that $Z^1(\Gamma, M)$ is a right M-module. Since every cocycle $c : \Gamma \to M$ is uniquely determined the by the values $c(a_1), ..., c(a_m)$ we have an isomorphism of right M-modules $\Phi : M^{\otimes m} \to Z^1(\Gamma, M)$ given by $\Phi(x_1, ..., x_m)$ is the unique cocycle $c : \Gamma \to M$ such that $c(a_1) = x_1, ..., c(a_m) = x_m$.

Next, let $(g_k)_k \subset \Gamma$ be group elements and $(A_k)_k \subset X$ be measurable sets such that $\mathcal{R}$ is the smallest equivalence relation on X containing $\cup_k \{(x, g_k \cdot x) \mid x \in A_k\}$. Put $p_k = 1_{g_k A_k} \in \mathcal{P}(M)$ and define $\varphi_k : A_k \to g_k \cdot A_k$ by letting $\varphi_k(x) = g_k \cdot x$.

We define a right M-modular map $\Psi : Z^1(\Gamma, M) \to \oplus_k p_k M$ by letting $\Psi(c) = \oplus_k p_k c(g_k)$. We claim that Ψ is injective. To prove the claim, let $c : \Gamma \to M$ be a cocycle such that $p_k c(g_k) = 0$, for all k. Then for every $k_1, .., k_n$ we have that $(p_{k_l} \circ (g_{k_1} ... g_{k_{l-1}})^{-1}) u_{g_{k_1}} ... u_{g_{k_{l-1}}} c(g_{k_l}) = 0$ and thus

$$\begin{aligned} & p_{k_1}(p_{k_2} \circ g_{k_1}^{-1}) ... (p_{k_n} \circ (g_{k_1} ... g_{k_{n-1}})^{-1}) c(g_{k_1} ... g_{k_n}) \\ & = \sum_{l=1}^{n} [p_{k_1}(p_{k_2} \circ g_{k_1}^{-1}) ... (p_{k_n} \circ (g_{k_1} ... g_{k_{n-1}})^{-1})] u_{g_{k_1}} ... u_{g_{k_{l-1}}} c(g_{k_l}) = 0 \end{aligned} \tag{14.1}$$

We are now ready to show that if $g \in \Gamma$, then $c(g) = 0$. Let $A \subset X$ be a non-null measurable set. Then we can find a non-null measurable subset $B \subset A$ and $k_1, .., k_n$ such that B is contained in the domain of $\varphi_{k_1} \circ ... \circ \varphi_{k_n}$ and $g \cdot x = (\varphi_{k_1} \circ ... \circ \varphi_{k_n})(x)$, for all $x \in B$. Equivalently $1_B \leq (1_{A_{k_1}} \circ (g_{k_2} ... g_{k_n})) ... (1_{A_{k_{n-1}}} \circ g_{k_n}) 1_{A_{k_n}}$ and $g = g_{k_1} ... g_{k_n}$. Thus, we have

$$\begin{aligned} 1_{gB} = 1_{g_{k_1} ... g_{k_n} B} &= 1_B \circ (g_{k_1} ... g_{k_n})^{-1} \\ &\leq (1_{A_{k_1}} \circ g_{k_1}^{-1}) ... (1_{A_{k_n}} \circ (g_{k_1} ... g_{k_n})^{-1}) \\ &= p_{k_1} ... (p_{k_n} \circ (g_{k_1} ... g_{k_{n-1}})^{-1}). \end{aligned}$$

This and (15.3) imply that $1_{gB} c(g) = 1_{gB} c(g_1 ... g_k) = 1_{gB} p_{k_1} ... (p_{k_n} \circ (g_{k_1} ... g_{k_{n-1}})^{-1}) c(g_1 ... g_k) = 0$. Thus, for every non-null set A we can find a non-null subset B such that $1_{gB} c(g) = 0$. This implies that $c(g) = 0$. Since $g \in \Gamma$ is arbitrary, we derive that $c = 0$ and thus Ψ is injective.

Therefore, $\Psi \circ \Phi : M^{\oplus m} \to \oplus_k p_k M$ is a right M-modular injective map. By Theorem 10.21, we conclude that $m \leq \sum_k \tau(p_k) = \sum_k \mu(A_k)$. ■

Proof of Theorem 14.1. Assume that for some $m > n$, we can find free ergodic pmp actions $\mathbb{F}_m \curvearrowright (X, \mu)$ and $\mathbb{F}_n \curvearrowright (Y, \nu)$ which are orbit equivalent. Let $\theta : X \to Y$ be an orbit equivalence.

If we consider the action $\mathbb{F}_n \curvearrowright (X,\mu)$ given by $g\cdot x = \theta^{-1}(g\cdot\theta(x))$, then $\mathbb{F}_m\cdot x = \mathbb{F}_n\cdot x$, for almost every $x\in X$. Put $\mathcal{R} := \{(x, g\cdot x) \mid x\in X, g\in\mathbb{F}_m\}$. Let $b_1,...,b_n\in\mathbb{F}_n$ be free generators. For every $1\leq i\leq n$ and $g\in\mathbb{F}_m$, let $A_{i,g} = \{x\in X \mid b_i\cdot x = g\cdot x\}$.

Then $\mathcal{R}$ is the smallest equivalence relation on X containing $\cup_{i=1}^n\{(x, a_i\cdot x) \mid x\in X\}$, and thus the smallest equivalence on X containing $\cup_{i=1}^n(\cup_{g\in\mathbb{F}_m}\{(x,g\cdot x) \mid x\in A_{i,g}\})$. Theorem 14.2 implies that

$$m \leq \sum_{i=1}^n\sum_{g\in\mathbb{F}_m}\mu(A_{i,g}) = \sum_{i=1}^n(\sum_{g\in\mathbb{F}_m}\mu(A_{i,g})) = \sum_{i=1}^n 1 = n,$$

which contradicts that $m > n$. ∎

15. Solidity of free group factors

Theorem 15.1 (Ozawa, 2003)**.** *Let $M = L(\mathbb{F}_n)$, for some $2\leq n\leq\infty$. Then M is* **solid***: for every diffuse von Neumann subalgebra $A\subset M$, the relative commutant $A'\cap M$ is amenable.*

We present here Popa's proof of this theorem [Po06] based on his powerful deformation/rigidity theory. Let us first explain the *deformation* property of M that is used in the proof.

For simplicity, we assume that $n = 2$. Denote by a_1, a_2, b_1, b_2 the generators of $\mathbb{F}_4$, and view $\mathbb{F}_2$ as the subgroup of $\mathbb{F}_4$ generated by a_1, a_2. This gives an embedding of M into $\tilde{M} = L(\mathbb{F}_4)$. We denote still by $a_,a_2, b_1, b_2$ the corresponding unitaries in $\tilde{M}$. Consider the argument function $\text{Arg}:\mathbb{T}\rightarrow[-\pi,\pi]$ and use Borel functional calculus to define $h_1 = \text{Arg}(b_1), h_2 = \text{Arg}(b_2)$. Then $h_1\in\{b_1\}''$, $h_2\in\{b_2\}''$ are self-adjoint operators such that $b_1 = \exp(ih_1)$ and $b_2 = \exp(ih_2)$. Moreover, for every $t\in\mathbb{R}$, we have that

$$\tau(\exp(ith_1)) = \tau(\exp(ith_2)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(it\theta)\,\mathrm{d}\theta = \frac{\sin(\pi t)}{\pi t}. \tag{15.1}$$

For every $t\in\mathbb{R}$, we can define a trace preserving automorphism θ_t of $\tilde{M}$ by letting:

$$\theta_t(a_1) = \exp(ith_1)a_1, \quad \theta_t(a_2) = \exp(ith_2)a_2, \quad \theta_t(b_1) = b_1, \quad \theta_t(b_2) = b_2. \tag{15.2}$$

It is easy to see that $\lim_{t\rightarrow 0}\|\alpha_t(x) - x\|_2 = 0$, for all $x\in\tilde{M}$. Denote $\rho(t) = \frac{\sin(\pi t)}{\pi t}$ and notice that $0 < \rho(t) < 1$, for all $t > 0$. We leave it as an exercise to show that formula (15.1) implies that:

Lemma 15.2. *If $t\in\mathbb{R}$, $g\in\mathbb{F}_2$, then $E_M(\alpha_t(u_g)) = \rho(t)^{|g|}u_g$. Here, $|g|$ denotes the* **word length** *of $g\in\mathbb{F}_2$ with respect to the generating set $\{a_1, a_2\}$.*

Corollary 15.3. *Let $(u_n)\subset\mathcal{U}(M)$ be a sequence converging weakly to 0: $\tau(u_n x)\rightarrow 0$, for all $x\in M$. Then $\lim_{n\rightarrow\infty}\|E_M(\alpha_t(u_n))\|_2 = 0$, for all $t > 0$.*

Proof. Consider the Fourier expansion $u_n = \sum_{g\in\mathbb{F}_2}\tau(u_n u_g^*)u_g$. Then by Lemma 15.2 we get that $E_M(\alpha_t(u_n)) = \sum_{g\in\mathbb{F}_2}\rho(t)^{|g|}\tau(u_n u_g^*)u_g$ and thus $\|E_M(\alpha_t(u_n))\|_2^2 = \sum_{g\in\mathbb{F}_2}\rho(t)^{2|g|}|\tau(u_n u_g^*)|^2$. Thus, if $N\geq 1$ is an integer, then using that $\sum_{g\in\mathbb{F}_2}|\tau(u_n u_g^*)|^2 = \|u_n\|_2^2 = 1$, we get that

$$\|E_M(\alpha_t(u_n))\|_2^2 \leq \sum_{|g|\leq N}|\tau(u_n u_g^*)|^2 + \rho(t)^{2N}\sum_{|g|\geq N}|\tau(u_n u_g^*)|^2 \leq \sum_{|g|\leq N}|\tau(u_n u_g^*)|^2 + \rho(t)^{2N}.$$

Since the set $\{|g|\leq N\}$ is finite and $\lim_{n\rightarrow\infty}\tau(u_n u_g^*) = 0$, for all $g\in\mathbb{F}_2$, we conclude that $\limsup_{n\rightarrow\infty}\|E_M(\alpha_t(u_n))\|_2^2 \leq \rho(t)^{2N}$, for all $N\geq 1$. Since $0 < \rho(t) < 1$, the conclusion follows. ∎

Lemma 15.4. *The Hilbert M-bimodule $L^2(\tilde{M})\ominus L^2(M)$ is isomorphic to an infinite multiple of the coarse M-bimodule, $(L^2(M)\bar{\otimes}L^2(M))^{\oplus\infty}$. Moreover, for every von Neumann subalgebra $B\subset M$, the Hilbert B-bimodule $L^2(\tilde{M})\ominus L^2(M)$ is isomorphic to $(L^2(B)\bar{\otimes}L^2(B))^{\oplus\infty}$.*

Proof. Let $S\subset\mathbb{F}_4$ be the set of elements $g\in\mathbb{F}_2$ whose reduced form begins and ends with a non-zero power of b_1 or b_2. Since $L^2(\tilde{M})\ominus L^2(M)=\bigoplus_{g\in S}\overline{Mu_gM}$, in order to prove the main assertion, it suffices to show that $\overline{Mu_gM}\cong L^2(M)\bar{\otimes}L^2(M)$, as Hilbert M-bimodules, for any $g\in S$.

If $g\in S$, then $g^{-1}\mathbb{F}_2g\cap\mathbb{F}_2=\{e\}$. Hence, $\tau(u_g^*u_hu_gu_k)=\delta_{g^{-1}hg,k^{-1}}=\delta_{h,e}\delta_{k,e}=\tau(u_h)\tau(u_k)$, for all $h,k\in\mathbb{F}_2$. This implies $\tau(u_g^*au_gb)=\tau(a)\tau(b)$, for all $a,b\in M$, and further that

$$\langle xu_gy,zu_gt\rangle=\tau(u_g^*z^*xu_gyt^*)=\tau(z^*x)\tau(yt^*)=\langle x\otimes y,z\otimes t\rangle_{L^2(M)\bar{\otimes}L^2(M)},\text{ for all } x,y,z,t\in M.$$

Thus, $x\otimes y\mapsto xu_gy$ extends to an isomorphism of Hilbert M-bimodules $L^2(M)\bar{\otimes}L^2(M)\cong\overline{Mu_gM}$.

If we let $\mathcal{H}=L^2(M)^{\oplus\infty}$, the main assertion implies that $L^2(\tilde{M})\ominus L^2(M)\cong\mathcal{H}\bar{\otimes}\mathcal{H}$, as M-bimodules. If $B\subset M$ is a von Neumann subalgebra, then $\mathcal{H}$ is an infinite dimensional left (and right) B-module and thus it is isomorphic to $L^2(B)^{\oplus\infty}$ as a left (and right) B-module. The moreover assertion is now immediate. ■

Proof of Theorem 15.1. Denote $B=A'\cap M$. Since A is diffuse, we can find a sequence $(u_k)_k\subset\mathcal{U}(A)$ which converges weakly to 0. Indeed, since A is diffuse, by Exercise 8.8 and Corollary 8.6 it contans a copy of $L^\infty([0,1])$. If $u_k\in\mathcal{U}(L^\infty([0,1]))$ is given by $u_k(x)=\exp(2\pi ikx)$, then $u_k\to 0$ weakly.

Put $t_n=1/2^n$, for every $n\geq 1$. By Corollary 15.3 we can find a subsequence $(v_n)_n$ of $(u_k)_k$ such that $\lim_{n\to\infty}\|E_M(\alpha_{t_n}(v_n))\|_2=0$. Define $\xi_n:=\alpha_{t_n}(v_n)-E_M(\alpha_{t_n}(v_n))$. We claim that

$$\lim_{n\to\infty}\|[x,\xi_n]\|_2=0,\text{ for every } x\in B,\tag{15.3}$$

$$\lim_{n\to\infty}\langle x\xi_n,\xi_n\rangle=\tau(x),\text{ for every } x\in\tilde{M}.\tag{15.4}$$

If $x\in B$, then by using that $[x,v_n]=0$ for all n and that $\alpha_t^{-1}=\alpha_{-t}$, we get that

$$\begin{aligned}\|[x,\xi_n]\|_2&=\|[x,\alpha_{t_n}(v_n)]-E_M([x,\alpha_{t_n}(v_n)])\|_2\\&\leq 2\|[x,\alpha_{t_n}(v_n)]\|_2\\&=2\|[\alpha_{-t_n}(x),v_n]\|_2\\&=2\|[\alpha_{-t_n}(x)-x,v_n]\|_2\\&\leq 2\|\alpha_{-t_n}(x)-x\|_2.\end{aligned}$$

Since $\lim_{t\to 0}\|\alpha_t(x)-x\|_2=0$, equation (15.3) follows. Since $\lim_{n\to\infty}\|E_M(\alpha_{t_n}(v_n))\|_2=0$, we get that $\lim_{n\to\infty}\langle x\xi_n,\xi_n\rangle=\lim_{n\to\infty}\langle x\alpha_{t_n}(v_n),\alpha_{t_n}(v_n)\rangle=\tau(x)$, for every $x\in\tilde{M}$, which proves (15.4).

We are now ready to prove that B is amenable. Since $\xi_n\in\tilde{M}$ and $E_M(\xi_n)=0$, we can view ξ_n as a vector in $L^2(\tilde{M})\ominus L^2(M)$. Formulae (15.3) and (15.4) assert that ξ_n are B-almost central and almost tracial. By Lemma 15.4 we have that $L^2(\tilde{M})\ominus L^2(M)$ is isomorphic to $L^2(B)\bar{\otimes}\mathcal{K}$ as a Hilbert B-bimodule, where $\mathcal{K}=L^2(B)^{\oplus\infty}$. Denote by $J:\mathcal{K}\to\mathcal{K}$ the involution given by $J(\oplus b_i)=\oplus b_i^*$. Then for every $x\in B$, $JxJ\in\mathbb{B}(\mathcal{K})$ is the right multiplication operator by x^*.

Let ω be a free ultrafilter on $\mathbb{N}$, and define a state $\Phi:\mathbb{B}(L^2(B))\to\mathbb{C}$ by letting

$$\Phi(T)=\lim_{n\to\omega}\langle(T\otimes\mathrm{Id}_{\mathcal{K}})\xi_n,\xi_n\rangle,\text{ for every } T\in\mathbb{B}(L^2(B)).$$

Then (15.4) implies that $\Phi_{|B} = \tau$. Moreover, if $x \in B$ and $T \in \mathbb{B}(L^2(B))$, by using (15.3) we get

$$\begin{aligned}
\Phi(xT) &\approx \langle (xT \otimes \mathrm{Id}_{\mathcal{K}})\xi_n, \xi_n\rangle \\
&= \langle (T \otimes \mathrm{Id}_{\mathcal{K}})\xi_n, (x^* \otimes \mathrm{Id}_{\mathcal{K}})\xi_n\rangle \\
&\approx \langle (T \otimes \mathrm{Id}_{\mathcal{K}})\xi_n, (\mathrm{Id}_{L^2(B)} \otimes JxJ)\xi_n\rangle \\
&\approx \langle (T \otimes \mathrm{Id}_{\mathcal{K}})(\mathrm{Id}_{L^2(B)} \otimes Jx^*J)\xi_n, \xi_n\rangle \\
&\approx \langle (T \otimes \mathrm{Id}_{\mathcal{K}})(x \otimes \mathrm{Id}_{\mathcal{K}})\xi_n, \xi_n\rangle \\
&= \langle (Tx \otimes \mathrm{Id}_{\mathcal{K}})\xi_n, \xi_n\rangle \\
&\approx \Phi(Tx),
\end{aligned}$$

where we write $x_n \approx y_n$ to mean that $\lim_{n\to\omega}(x_n - y_n) = 0$.

This shows that B is amenable and finishes the proof. ■

Exercise 15.5. Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras.

(1) Prove that $M_1 \bar{\otimes} M_2$ is a tracial von Neumann algebra.
(2) Prove that if M_1 and M_2 are amenable, then $M_1 \bar{\otimes} M_2$ is amenable.

Exercise 15.6. Let M be a solid II_1 factor. Prove that if M is not amenable, then M is **prime**, i.e., it cannot be written as a tensor product $M = M_1 \bar{\otimes} M_2$, for some II_1 factors M_1 and M_2.

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